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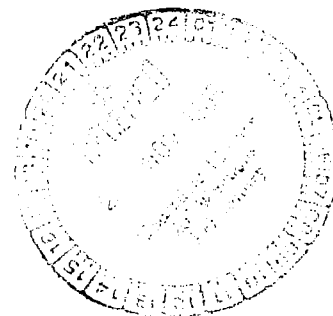


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EQUATIONS FOR THERMOELASTIC  
AND VISCOELASTIC  
CYLINDRICAL SANDWICH SHELLS

*by Wolfgang J. Oberndorfer*

*Prepared by*  
UNIVERSITY OF CALIFORNIA  
Berkeley, Calif.  
*for*





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### ABSTRACT

Equations governing small deformation of a cylindrical Sandwich Shell are derived from the principle of Minimum Potential Energy. The facings are thin orthotropic shells with different physical properties and thicknesses, and a weak orthotropic core is considered. The effect of an arbitrary temperature distribution is included. Two examples are presented: (1) uniform heating of a finite segment of a shell of infinite extent (elastic core) and (2) uniform heating of a shell of infinite extent (viscoelastic core) supporting a ring load.

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## 1. Introduction

A recent survey by Habip [1] of developments in the analysis of sandwich structures, including shells, has disclosed that rather limited attention has been paid to thermoelastic or viscoelastic shell problems.

Although the theory of sandwich structures is dated from at least 1940, the temperature effect apparently was not considered until 1959. At that time a paper by Bijlaard [2] was published; he assumed a constant temperature gradient between the facings, found the equivalent moment and calculated the corner forces for a simply supported rectangular plate, dealing only with the gross magnitudes of the plate. No stress calculations and no constitutive equations are shown. The usual Reissner sandwich plate assumptions were made, taking into account the shear in the core and considering only small deflections. Yao [3] considered a cylinder of 3 thin layers (all 3 considered as membranes), the inner one having different elastic properties. Applying the Cross iteration procedure he achieved compatibility of the displacements between the layers and calculated the resulting deformations. Since moments are not considered in this method it must be regarded as a very rough approximation.

Kendall et al., [4] proposed a method for predicting thermal response in sandwich plates and a few additional references may be found in [1].

A treatment of the thermoelastic equations for sandwich plates appears in the work of Chang and Ebcioğlu [5]; assuming the Duhamel-Neumann stress-strain law they derived differential equations for sandwich plates from the principle of Minimum Potential Energy. Further assumptions are:

small deflections, facings are isotropic membranes, the orthotropic core transfers only transverse shear stresses and is incompressible in the normal direction. Chang's results of compression tests on sandwich plates with facings kept at different temperatures are published in [6], but no mathematical treatment is attached. In [7] Ebcioğlu repeated the equations of [5] by means of the usual kinematic and equilibrium considerations but included an arbitrary temperature distribution. He took into account transverse normal strain and surface stress couples and arrived at a complete set of equations for the thermoelastic behavior of a sandwich plate. In [8] the above theory was extended for large deflections and rotations. This paper constitutes the most general treatment on sandwich plates available at present. The following three papers dealing with plates were inaccessible to the author: in [9] a general treatment of sandwich plates is given, and [10] and [11] deal with the thermoelastic vibrations of a sandwich plate.

A work of generality comparable to [8] for shells has not appeared. The papers [12] by Grigoliuk, [13] by Vasitsina and [14] by Mushtari are on the level of [5] and [7] but are limited to shallow shells with constant transverse normal displacement in the core.

The viscoelastic behavior of a sandwich plate is treated for the first time in [15]: from a generalized Hamilton's principle Chang obtained the three differential equations of motion in the Laplace space; only vibrations are studied, no load cases. The ordinary Reissner sandwich plate assumptions are made together with small deflections.

Yu [16] studied equations for the frequency of damped vibrations of a sandwich plate in great generality, without showing how the concept of viscoelastic damping fits his assumption of perfect elastic materials.

Baylor [17] presented a general theory of anisotropic viscoelastic sandwich shells in coordinate invariant tensor notation and took into account kinematic refinements but limited to small deflections. As an example he investigated the infinite circular cylinder with viscoelastic core under a ring load. Other than this example, no treatment of sandwich structures with a viscoelastic core could be found.

In Section 2 the principle of Minimum Potential Energy is used to obtain the displacement equations of equilibrium for cylindrical shells. The derivation follows the procedure given in [5] and the results are presented as in [16], although the 3-term expansion for the transverse displacement, the orthotropy of the material, and the inclusion of the temperature gradient makes the variation more complicated. The core material is allowed to be viscoelastic with temperature dependent material properties. Only small deflections are considered; strains and rotations are small compared with unity, and the following expressions for the displacements are used:

$$\begin{aligned}\underline{u}(x, \theta, z) &= u(x, \theta) + z \phi(x, \theta), \\ \underline{v}(x, \theta, z) &= v(x, \theta) + z \chi(x, \theta), \\ \underline{w}(x, \theta, z) &= w(x, \theta) + z \psi(x, \theta) + \frac{z^2}{2} \chi(x, \theta).\end{aligned}\tag{1}$$



Surface loads and couples are also considered in the equilibrium equations. The displacement assumption (1)<sub>3</sub> generalizes the work reported in [12], [13], [14] in the sense that core compressibility is permitted, although large displacements are excluded.

In Section 3 examples show the deflection under a spatial distribution of temperature for a sandwich cylinder and the deflection under uniform heating for a sandwich cylinder carrying a ringload. Solutions are obtained by means of integral transforms and a viscoelastic core is considered in the second example.

The principle of Minimum Potential Energy is needed in the sequel and appears here for reference:

$$I = \iiint_V \left( \frac{1}{2} \epsilon_{ij} \tau_{ij} + \delta_{ij} \alpha_{ik} \tau_{jk} T \right) dV - \iint_{S_1} \bar{T}_i u_i dA, \quad (2)$$

$$\delta I = 0.$$

The first integral contains the strain energy in terms of the displacements and temperature; variation with respect to the displacements yields the displacement equations of equilibrium. The second integral consists of the integral  $W_1$  over the surface and the integral  $W_2$  along the edge:

$$\iint_{S_1} \bar{T}_i u_i dA = \underbrace{\iint \bar{P}_i u_i dA}_{W_1} + \underbrace{\int d\mathbf{z} \oint \bar{\tau}_{ij} n_i u_j ds}_{W_2} \quad (3)$$

Variation of  $W_1$  yields the load terms, variation of  $W_2$  the natural boundary conditions for the shell stress resultants.

## 2. Equations for an Elastic Cylindrical Sandwich Shell

### 2.1) The contribution of the facings to the strain energy and to the Euler equations

The shell facings are assumed to be in a state of generalized plane stress. In this case the constitutive law for an orthotropic material [18] is:

$$\begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \cdot \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \\ 2\epsilon_{12} \end{bmatrix} \quad (4)$$

where:

$$C_{13} = 0$$

$$C_{23} = 0$$

$$C_{33} = 0$$

$$C_{44} = 0$$

$$C_{55} = 0$$

$$C_{11} = \frac{E_1}{1-\nu_1\nu_2}, \quad C_{12} = \frac{E_1\nu_2}{1-\nu_1\nu_2}, \quad C_{21} = \frac{E_2\nu_1}{1-\nu_1\nu_2}, \quad (5)$$

$$C_{22} = \frac{E_2}{1-\nu_1\nu_2}, \quad C_{66} = G.$$

Introducing temperature change  $T$  through the Duhamel-Neumann constitutive law, we have

$$\begin{aligned}\epsilon_{11} &= \frac{1}{E_1} \tau_{11} - \frac{\nu_2}{E_2} \tau_{22} + \alpha_1 T, \\ \epsilon_{22} &= \frac{1}{E_2} \tau_{22} - \frac{\nu_1}{E_1} \tau_{11} + \alpha_2 T, \\ \epsilon_{12} &= \frac{1}{2G} \tau_{12}.\end{aligned}\tag{6}$$

Accordingly, the stress-strain relations including the temperature strain for an orthotropic material can be written

$$\begin{aligned}\tau_{11} &= \frac{E_1}{1-\nu_1\nu_2} \left[ (\epsilon_{11} + \nu_2 \epsilon_{22}) - (\alpha_1 + \nu_2 \alpha_2) T \right], \\ \tau_{22} &= \frac{E_2}{1-\nu_1\nu_2} \left[ (\epsilon_{22} + \nu_1 \epsilon_{11}) - (\alpha_2 + \nu_1 \alpha_1) T \right], \\ \tau_{12} &= 2G \epsilon_{12}.\end{aligned}\tag{7}$$

Using the abbreviations  $(\alpha_1 + \nu_2 \alpha_2) T = \alpha T_1$ ,

and  $\underbrace{\frac{E_2 \nu_1}{E_1 \nu_2}}_1 (\alpha_2 + \nu_1 \alpha_1) T = \alpha T_2$ ,

the expression for the strain energy per unit volume in generalized plane stress becomes

$$\begin{aligned}\frac{1}{2} \epsilon_{\alpha\beta} \tau_{\alpha\beta} + \delta_{ij} \alpha_i \beta_j \tau_{ij} T &= \frac{1}{2} \frac{E_1}{1-\nu_1\nu_2} (\epsilon_{11}^2 + 2\nu_2 \epsilon_{11} \epsilon_{22} + \\ &+ \frac{\nu_2}{\nu_1} \epsilon_{22}^2 - 2\alpha T_1 \epsilon_{11} - 2\frac{\nu_2}{\nu_1} \alpha T_2 \epsilon_{22}) \\ &+ 2G \epsilon_{12}^2.\end{aligned}\tag{8}$$

The strain-displacement relations in cylindrical coordinates are

$$\begin{aligned}\epsilon_{11} &= u_{,x} , \\ \epsilon_{22} &= \frac{1}{a} (v_{,\theta} + w) , \\ \epsilon_{12} &= \frac{1}{2} (v_{,x} + \frac{1}{a} u_{,\theta}) .\end{aligned}\tag{9}$$

We denote all quantities of the upper facing by a prime and all quantities of the lower facing by two primes. After having introduced (9) in (8) we obtain the total strain energy of the facings by integrating over the thicknesses  $h'$  and  $h''$ , summing, and integrating over the surface:

$$\begin{aligned}U_f' &= \frac{h'}{2} \left\{ \frac{E'}{1-\nu_1'\nu_2'} \left[ u_{,x}'^2 + 2\nu_2' u_{,x}' + \frac{1}{a'} (v_{,\theta}' + w') + \right. \right. \\ &+ \left. \frac{\nu_2'}{\nu_1'} \frac{1}{(a')^2} (v_{,\theta}' + w')^2 + 2\alpha'T_1' u_{,x}' + 2\frac{\nu_2'}{\nu_1'} \alpha'T_2 \frac{1}{a'} (v_{,\theta}' + w') \right] \\ &+ \left. G (v_{,x}' + \frac{1}{a'} u_{,\theta}')^2 \right\} ,\end{aligned}\tag{10}$$

$$U = \int_{x=0}^L \int_{\theta=0}^{\alpha} (u_f' a' + u_f'' a'') dx d\theta .\tag{11}$$

Using (1), continuity of displacements at interfaces requires that

$$\begin{aligned}u' &= u + \frac{h}{2} \phi , \quad v' = v + \frac{h}{2} \chi , \quad w' = w + \frac{h}{2} \psi + \frac{h^2}{8} \kappa , \\ u'' &= u - \frac{h}{2} \phi , \quad v'' = v - \frac{h}{2} \chi , \quad w'' = w - \frac{h}{2} \psi + \frac{h^2}{8} \kappa .\end{aligned}\tag{12}$$

Considering (10) and (12) we find from (11):

$$\begin{aligned}
 U = \iint & \left\{ \frac{E'h' a'_a}{2(1-\nu'_1\nu'_2)} \left[ (u_{,x} + \frac{h}{2} \phi_{,x})^2 + 2\nu'_2 (u_{,x} + \frac{h}{2} \phi_{,x}) \frac{1}{a'} (v_{,\theta} + \right. \right. \\
 & + \frac{h}{2} \chi_{,\theta} + \omega + \frac{h}{2} \psi + \frac{h^2}{8} \chi) + \frac{\nu'_2}{\nu'_1} \frac{1}{a'^2} (v_{,\theta} + \frac{h}{2} \chi_{,\theta} + \omega + \frac{h}{2} \psi + \frac{h^2}{8} \chi)^2 \\
 & - 2\alpha' \tau'_1 (u_{,x} + \frac{h}{2} \phi_{,x}) - 2\frac{\nu'_2}{\nu'_1} \alpha' \tau'_2 \frac{1}{a'} (v_{,\theta} + \frac{h}{2} \chi_{,\theta} + \omega + \frac{h}{2} \psi + \frac{h^2}{8} \chi) \Big] + \\
 & + \frac{G'h' a'_a}{2} \left[ v_{,x} + \frac{h}{2} \chi_{,x} + \frac{1}{a'} (u_{,\theta} + \frac{h}{2} \phi_{,\theta}) \right]^2 + \frac{E''h'' a''_a}{2(1-\nu''_1\nu''_2)} \left[ (u_{,x} + \right. \\
 & - \frac{h}{2} \phi_{,x})^2 + 2\nu'_2 (u_{,x} - \frac{h}{2} \phi_{,x}) \frac{1}{a''} (v_{,\theta} - \frac{h}{2} \chi_{,\theta} + \omega - \frac{h}{2} \psi + \frac{h^2}{8} \chi) + \\
 & + \frac{\nu''_2}{\nu''_1} \frac{1}{a''^2} (v_{,\theta} - \frac{h}{2} \chi_{,\theta} + \omega - \frac{h}{2} \psi + \frac{h^2}{8} \chi)^2 - 2\alpha'' \tau''_1 (u_{,x} - \frac{h}{2} \phi_{,x}) + \\
 & - 2\frac{\nu''_2}{\nu''_1} \alpha'' \tau''_2 \frac{1}{a''} (v_{,\theta} - \frac{h}{2} \chi_{,\theta} + \omega - \frac{h}{2} \psi + \frac{h^2}{8} \chi) \Big] + \frac{G''h'' a''}{2} \cdot \\
 & \cdot \left[ v_{,x} - \frac{h}{2} \chi_{,x} + \frac{1}{a''} (u_{,\theta} - \frac{h}{2} \phi_{,\theta}) \right]^2 \Big\} a \, dx \, d\theta \quad (13)
 \end{aligned}$$

This result agrees with the linearized expression given in [24] for a homogeneous shell.

A short remark should be made on the displacement assumptions: the 2-term assumption for  $\underline{u}$  and  $\underline{v}$  enables us to include shear effects in the core and we are on the same level as Reissner's refined theory. The Kirchhoff-Love assumption  $\phi = -w_{,x}$  and  $\chi = -w_{,\theta}$  does not allow this refinement. The 3-term assumption for the transverse displacement enables us to study a linear variation of the normal strain and introduces constitutive equations for the transverse shear and normal stresses. The importance of this type of transverse displacement assumption for sandwich shells has been mentioned by Reissner [25].

The Euler equation associated with a typical variable  $u$  in the above is:

$$\delta_u \equiv U_{,u} - \frac{\partial}{\partial x} U_{,u,x} - \frac{1}{a} \frac{\partial}{\partial \theta} U_{,u,\theta} = 0 \quad (14)$$

Thus, we obtain seven partial differential equations (in part) by variations of the strain energy (13) with respect to  $u, v, w, \phi, \chi, \psi$ , and  $\lambda$ . It is convenient to define:

$$\begin{aligned} A_{1,2} &= 1 \pm \frac{E''h''(1-\nu_1'\nu_2')}{E'h'(1-\nu_1''\nu_2'')} , & T_{1,2} &= \alpha T_1' \pm \frac{E''h''(1-\nu_1'\nu_2')}{E'h'(1-\nu_1''\nu_2'')} \alpha'' T_1'' , \\ B_{1,2} &= \nu_2' \pm \nu_2'' \frac{E''h''(1-\nu_1'\nu_2')}{E'h'(1-\nu_1''\nu_2'')} , & T_{2,22} &= \frac{\nu_2'}{\nu_1'} \alpha_1 T_1' \pm \frac{\nu_2''}{\nu_1''} \alpha'' T_1'' . \\ C_{1,2} &= \frac{\nu_2'}{\nu_1'} \pm \frac{\nu_2''}{\nu_1''} \frac{E''h''(1-\nu_1'\nu_2')}{E'h'(1-\nu_1''\nu_2'')} , & & \frac{E''h''(1-\nu_1'\nu_2')}{E'h'(1-\nu_1''\nu_2'')} , \\ G_{1,2} &= 2(1-\nu_1'\nu_2') \left( \frac{\zeta_1'}{E'} \pm \frac{\zeta_1''h''}{E'h'} \right) , & \frac{h}{a} &= \tau . \end{aligned} \quad (15)$$

to factor out  $E'h'/2(1-\nu_1'\nu_2')$  from all equations and to use the expansions:

$$\frac{a'}{a} = 1 + \frac{\tau}{2} , \quad \frac{a''}{a} = 1 - \frac{\tau}{2} , \quad \frac{a}{a'} = 1 - \frac{\tau}{2} + \frac{\tau^2}{4} , \quad \frac{a}{a''} = 1 + \frac{\tau}{2} + \frac{\tau^2}{4} .$$

After some lengthy computations and comparisons it was found that the following procedure must be adopted in order to be able to reduce to Flügge's differential equations of a homogeneous cylindrical shell:  $\tau^2/4$  must not be neglected in comparison with 1 if it stands with  $u, v, w$  or their derivatives, but it is neglected if it stands with  $\phi, \chi$  or their derivatives. The differential equations of a homogeneous cylindrical shell according to Vlasov are obtained if  $\tau^2/4$  is neglected in comparison with 1 if it does

not stand with w. The Donnell accuracy will be reached by neglecting  $\tau/2$  and  $\tau^2/4$  compared with 1 consistently if it does not stand with w. Retaining the Flügge order of approximation the contributions of the facings to the Euler equations are:

$$\begin{aligned} \delta u = & -2(A_1 + \tau A_2) u_{,xx} - \left[ \left(1 + \frac{\tau^2}{4}\right) G_1 - \frac{\tau}{2} G_2 \right] \frac{1}{a^2} u_{,\theta\theta} - (2B_1 + G_1) \frac{1}{a} v_{,\theta x} - \tau(A_2 + \\ & + \frac{\tau}{2} A_1) a \phi_{,xx} - \frac{\tau}{2} (G_2 - \frac{\tau}{2} G_1) \frac{1}{a} \phi_{,\theta\theta} - \frac{\tau}{2} (2B_2 + G_2) \chi_{,x\theta} - \\ & - \tau B_2 \psi_{,x} - \frac{\tau^2}{4} B_2 a \chi_{,x} + 2(T_{11} + \tau/2 T_{12})_{,x} \end{aligned} \quad (16)$$

$$\begin{aligned} \delta v = & -(2B_1 + G_1) \frac{1}{a} u_{,\theta} - (G_1 + \frac{\tau}{2} G_2) v_{,xx} - \left[ 2\left(1 + \frac{\tau^2}{4}\right) C_1 - \tau C_2 \right] \frac{1}{a^2} v_{,\theta\theta} \\ & - \frac{\tau}{2} (2B_2 + G_2) \phi_{,x\theta} - \frac{\tau}{2} (G_2 + \frac{\tau}{2} G_1) a \chi_{,xx} - \tau(C_2 - \frac{\tau}{2} C_1) \frac{1}{a} \chi_{,\theta\theta} - \left[ 2\left(1 + \frac{\tau^2}{4}\right) C_1 - \right. \\ & \left. - \tau C_2 \right] \frac{1}{a^2} \omega_{,\theta} + \frac{\tau^2}{2} C_1 \frac{1}{a} \psi_{,\theta} - \frac{\tau^2}{4} C_1 \chi_{,\theta} + \frac{2}{a} T_{21,\theta} \end{aligned} \quad (17)$$

$$\begin{aligned} \delta w = & 2B_1 \frac{1}{a} u_{,x} + \left[ 2\left(1 + \frac{\tau^2}{4}\right) C_1 - \tau C_2 \right] \frac{1}{a^2} v_{,\theta} + \tau B_2 \phi_{,x} + \tau \left( C_2 - \frac{\tau}{2} C_1 \right) \cdot \\ & \cdot \frac{1}{a} \chi_{,\theta} + 2 \left[ \left(1 + \frac{\tau^2}{4}\right) C_1 - \frac{\tau}{2} C_2 \right] \frac{1}{a^2} \omega + \tau \left( C_2 - \frac{\tau}{2} C_1 \right) \frac{1}{a} \psi + \\ & + \frac{\tau^2}{4} C_1 \chi - \frac{2}{a} T_{21} \end{aligned} \quad (18)$$

$$\begin{aligned} \delta \phi = & -\tau(A_2 + \frac{\tau}{2} A_1) u_{,xx} - \frac{\tau}{2} \left[ \left(1 + \frac{\tau^2}{4}\right) G_2 - \frac{\tau}{2} G_1 \right] \frac{1}{a^2} u_{,\theta\theta} - \frac{\tau}{2} (2B_2 + \\ & + G_2) \frac{1}{a} v_{,x\theta} - \frac{\tau^2}{2} (A_1 + \frac{\tau}{2} A_2) a \phi_{,xx} - \frac{\tau^2}{4} (G_1 - \frac{\tau}{2} G_2) \frac{1}{a} \phi_{,\theta\theta} - \\ & - \frac{\tau^2}{4} (2B_1 + G_1) \chi_{,\theta x} - \tau B_2 \frac{1}{a} \omega_{,x} - \frac{\tau^2}{2} B_1 \psi_{,x} - \frac{\tau^3}{8} B_2 a \chi_{,x} + \\ & + \tau(T_{12} + \frac{\tau}{2} T_{11})_{,x} \end{aligned} \quad (19)$$

$$\begin{aligned}
\delta \chi = & -\frac{\tau}{2} (2B_2 + G_2) \frac{1}{a} u_{,x\theta} - \frac{\tau}{2} (G_2 + \frac{\tau}{2} G_1) v_{,xx} - \tau \left[ (1 + \frac{\tau^2}{4}) C_2 + \right. \\
& \left. - \frac{\tau}{2} C_1 \right] \frac{1}{a^2} v_{,\theta\theta} - \frac{\tau^2}{4} (2B_1 + G_1) \phi_{,x\theta} - \frac{\tau^2}{4} (G_1 + \frac{\tau}{2} G_2) a \chi_{,xx} + \\
& - \frac{\tau^2}{2} (C_1 - \frac{\tau}{2} C_2) \frac{1}{a} \chi_{,\theta\theta} - \tau \left[ (1 + \frac{\tau^2}{4}) C_2 - \frac{\tau}{2} C_1 \right] \frac{1}{a^2} \omega_{,\theta} + \\
& - \frac{\tau^2}{2} (C_1 - \frac{\tau}{2} C_2) \frac{1}{a} \psi_{,\theta} - \frac{\tau^3}{8} C_2 \chi_{,\theta} + \frac{\tau}{a} \tau_{21,\theta}
\end{aligned} \tag{20}$$

$$\begin{aligned}
\delta \psi = & \tau B_2 \frac{1}{a} u_{,x} + \tau \left[ (1 + \frac{\tau^2}{4}) C_2 - \frac{\tau}{2} C_1 \right] \frac{1}{a^2} v_{,\theta} + \frac{\tau^2}{2} B_1 \phi_{,x} + \\
& + \frac{\tau^2}{2} (C_1 - \frac{\tau}{2} C_2) \frac{1}{a} \chi_{,\theta} + \tau \left[ (1 + \frac{\tau^2}{4}) C_2 - \frac{\tau}{2} C_1 \right] \frac{\omega}{a^2} + \\
& + \frac{\tau^2}{2} (C_1 - \frac{\tau}{2} C_2) \frac{1}{a} \psi + \frac{\tau^3}{8} C_2 \chi - \frac{1}{a} \tau \tau_{22}
\end{aligned} \tag{21}$$

$$\begin{aligned}
\delta \chi = & \frac{\tau^2}{4} B_1 \frac{1}{a} u_{,x} + \frac{\tau^2}{4} \left[ (1 + \frac{\tau^2}{4}) C_1 - \frac{\tau}{2} C_2 \right] \frac{1}{a^2} v_{,\theta} + \frac{\tau^3}{8} B_2 \phi_{,x} + \\
& + \frac{\tau^3}{8} (C_2 - \frac{\tau}{2} C_1) \frac{1}{a} \chi_{,\theta} + \frac{\tau^2}{4} \left[ (1 + \frac{\tau^2}{4}) C_1 - \frac{\tau}{2} C_2 \right] \frac{\omega}{a^2} + \\
& + \frac{\tau^3}{8} (C_2 - \frac{\tau}{2} C_1) \frac{1}{a} \psi + \frac{\tau^4}{32} C_1 \chi - \frac{1}{a} \frac{\tau^2}{4} \tau_{21}
\end{aligned} \tag{22}$$

## 2.2) The Contribution of the Core to the Strain Energy and to the Euler Equations

The core is assumed to transfer only transverse shear and normal stresses. Upon setting the in-plane stresses equal to zero we have the material constants

$$G_x = C_{44}$$

$$G_\theta = C_{55}$$

$$E_z = C_{33}$$



and the constitutive law

$$\begin{aligned}\tau_{13} &= 2 G_x \epsilon_{13}, \\ \tau_{23} &= 2 G_\theta \epsilon_{23}, \\ \tau_{33} &= E_z (\epsilon_{33} - \alpha_c T).\end{aligned}\tag{23}$$

The strain energy per unit volume in the core can be written as

$$\begin{aligned}U_c &= \frac{1}{2} \tau_{13} \epsilon_{13} + \delta_{ij} \alpha_{ij} \tau_{j3} = \frac{1}{2} \left[ 4 G_x \epsilon_{13}^2 + \right. \\ &\quad \left. + 4 G_\theta \epsilon_{23}^2 + E_z (\epsilon_{33} - 2\alpha_c T) \epsilon_{33} \right].\end{aligned}\tag{24}$$

We draw the strain-displacement relations from [24]:

$$\begin{aligned}\epsilon_{13} &= \frac{1}{2} (\phi + \omega_{,x} + z \psi_{,x} + \frac{z^2}{2} \chi_{,x}), \\ \epsilon_{23} &= \frac{1}{2} \left[ \chi + \frac{1}{a+z} (\omega_{,\theta} + z \psi_{,\theta} + \frac{z^2}{2} \chi_{,\theta} - \nu - z\chi) \right], \\ \epsilon_{33} &= \psi + z\chi.\end{aligned}\tag{25}$$

The total strain energy in the core is the integral over the volume:

$$\begin{aligned}U &= \frac{1}{2} \iiint_V \left\{ G_x (\phi + \omega_{,x} + z \psi_{,x} + \frac{z^2}{2} \chi_{,x})^2 + G_\theta \left[ \chi + \right. \right. \\ &\quad \left. \left. + \frac{1}{a+z} (-\nu + \omega_{,\theta} + z \psi_{,\theta} - z\chi + \frac{z^2}{2} \chi_{,\theta}) \right]^2 + E_z (\psi + \right. \\ &\quad \left. + z\chi - 2\alpha_c T)(\psi + z\chi) \right\} (1 + \frac{z}{a}) a dx d\theta dz = \\ &= \frac{1}{2} \iiint \left[ G_x h \left\{ (\phi + \omega_{,x})^2 + \frac{z^2}{12} a^2 [\psi_{,x}^2 + 2(\phi + \omega_{,x}) (\frac{1}{a} \psi_{,x} + \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \chi_{,x})] \right\} + G_\theta h \left\{ (\chi - \frac{\nu}{a} + \frac{\omega_{,\theta}}{a})^2 + \frac{z^2}{12} [(\psi_{,\theta} - \chi + \right. \right. \\ &\quad \left. \left. + \frac{\nu - \omega_{,\theta}}{a})^2 + a(\chi - \frac{\nu - \omega_{,\theta}}{a}) \chi_{,\theta}] \right\} + E_z h [\psi^2 - 2\psi T^{(1)} \right. \\ &\quad \left. - a \frac{z}{2} (\psi/a + \chi) T^{(2)} + \frac{z^2}{6} \chi (a\psi + \frac{a^2}{2} \chi - T^{(3)}) \right] a dx d\theta \end{aligned}\tag{26}$$

where these definitions were used in (26)

$$\begin{aligned} \int_{-\frac{h}{2}}^{\frac{h}{2}} \alpha_c T dz &= h T^{(1)}, & \int_{-\frac{h}{2}}^{\frac{h}{2}} \alpha_c T z dz &= \frac{h^2}{4} T^{(2)} \\ \int_{-\frac{h}{2}}^{\frac{h}{2}} \alpha_c T z^2 dz &= \frac{h^3}{12} T^{(3)} \end{aligned} \quad (27)$$

Variation of (26) with respect to the displacements, defining

$$\begin{aligned} D &= \frac{2 G_x h (1 - \nu_1' \nu_2')}{E' h'}, \\ E &= \frac{2 G_\theta h (1 - \nu_1' \nu_2')}{E' h'}, \\ F &= \frac{E_z h (1 - \nu_1' \nu_2')}{E' h'}, \end{aligned} \quad (28)$$

and multiplying all equations by  $\frac{2 (1 - \nu_1' \nu_2')}{E' h'}$ ,

the contributions of the core to the Euler equations are found to be:

$$\delta u \equiv 0 \quad (29)$$

$$\delta v = E \left[ \left( 1 + \frac{\tau^2}{12} \right) \frac{1}{a^2} (v, \omega, \theta) + \frac{\tau^2}{12} \left( \frac{1}{a} \psi, \theta - \frac{1}{2} u, \theta \right) - \frac{1}{a} \chi \right] \quad (30)$$

$$\begin{aligned} \delta \omega = D \left[ -(\phi, x + \omega, xx) - \frac{\tau^2}{12} a^2 \left( \frac{1}{a} \psi, xx + \frac{1}{2} \chi, xx \right) \right] + E \left[ -\left( 1 + \frac{\tau^2}{12} \right) \frac{1}{a^2} \cdot \right. \\ \left. \cdot (\omega, \theta\theta - v, \theta) + \frac{\tau^2}{12} \left( \frac{1}{a} \psi, \theta\theta - \frac{1}{2} \chi, \theta\theta \right) - \frac{1}{a} \chi, \theta \right] \end{aligned} \quad (31)$$

$$\delta \phi = D \left[ \frac{1}{a} (\omega, x + \phi) + \frac{\tau^2}{12} \left( \psi, x + \frac{a}{2} \chi, x \right) \right] \quad (32)$$

$$\delta \chi = E \left[ \left( 1 + \frac{\tau^2}{12} \right) \frac{1}{a^2} (\omega, \theta - v) - \frac{\tau^2}{12} \left( \frac{1}{a} \psi, \theta - \frac{1}{2} \chi, \theta \right) + \frac{1}{a} \chi \right] \quad (33)$$

$$\begin{aligned} \delta \psi = -D \frac{\tau^2}{12} (\omega, xx + \phi, x + a \psi, xx) - E \frac{\tau^2}{12} \frac{1}{a^2} (v, \theta - \omega, \theta\theta - \\ - a \chi, \theta + a \psi, \theta\theta) + 2F \left[ \frac{1}{a} (\phi - T^{(1)}) - \frac{\tau}{4} \frac{1}{a} T^{(2)} + \frac{\tau^2}{12} \chi \right] \end{aligned} \quad (34)$$

$$\delta \chi = -D \frac{\tau^2}{24} (\omega_{,xx} + \phi_{,x}) - E \frac{\tau^2}{24} \frac{1}{a^2} (\omega_{,\theta\theta} - \nu_{,\theta} + a \chi_{,\theta}) + 2F \left[ \frac{\tau}{4} \frac{1}{a} T^{(2)} + \frac{\tau^2}{12} \frac{1}{a} (\psi + a \chi - T^{(3)}) \right] \quad (35)$$

### 2.3) The Contribution of the External Work to the Euler Equations and the Resulting Displacement Equations of Equilibrium

The variation of the work of the external forces on the cylindrical boundary with respect to the displacements contributes the load terms:

$$-W_1 = \int_{x=0}^{\ell} \int_{\theta=0}^{\alpha} \bar{T}_i u_i a dx d\theta \quad \bar{T}_i = \text{prescribed traction on the cylindrical surface}$$

In this case the prescribed tractions are the external loads that are applied on the upper and lower facing. The work reads:

$$\begin{aligned} -W_1 = - \int \int \left\{ [P_x' (u + \frac{h}{2} \phi) + P_\theta' (v + \frac{h}{2} \chi) + P_z' (\omega + \frac{h}{2} \psi + \frac{h^2}{8} \chi)] \cdot \right. \\ \cdot (1 + \frac{\tau}{2}) + [P_x'' (u - \frac{h}{2} \phi) + P_\theta'' (v - \frac{h}{2} \chi) + P_z'' (\omega - \frac{h}{2} \psi + \\ \left. + \frac{h^2}{8} \chi)] (1 - \frac{\tau}{2}) \right\} a dx d\theta \end{aligned} \quad (36)$$

We now define surface load resultants and surface couples in the following manner:

$$\begin{aligned} P_x' (1 + \frac{\tau}{2}) + P_x'' (1 - \frac{\tau}{2}) = P_x, \quad [P_x' (1 + \frac{\tau}{2}) - P_x'' (1 - \frac{\tau}{2})] \frac{h}{2} = m_x, \\ P_\theta' (1 + \frac{\tau}{2}) + P_\theta'' (1 - \frac{\tau}{2}) = P_\theta, \quad [P_\theta' (1 + \frac{\tau}{2}) - P_\theta'' (1 - \frac{\tau}{2})] \frac{h}{2} = m_\theta, \\ P_z' (1 + \frac{\tau}{2}) + P_z'' (1 - \frac{\tau}{2}) = P_z, \quad [P_z' (1 + \frac{\tau}{2}) - P_z'' (1 - \frac{\tau}{2})] \frac{h}{2} = m_z. \end{aligned} \quad (37)$$

Upon introducing the symbols

$$\frac{2(1-\gamma_1'\nu_2')}{E'h'} P_i = P_i, \quad \frac{2(1-\gamma_1'\nu_2')}{E'h'} m_i = M_i \quad (38)$$

and upon taking the variation  $\delta W_1=0$  we obtain the load terms:

$$-P_x, -P_\theta, -P_z, -\frac{1}{a} M_x, -\frac{1}{a} M_\theta, -\frac{1}{a} M_z, -\frac{\tau^3}{8} P_z. \quad (39)$$

On the next pages the final equations which follow by summing up (16)-(22),

(29)-(35) and (39) are given:

$$\begin{aligned} \delta u: & (2A_1 + \tau A_2) u_{,xx} + \left[ (1 + \frac{\tau^2}{4}) G_1 - \frac{\tau}{2} G_2 \right] \frac{1}{a^2} u_{,\theta\theta} + (2B_1 + G_1) \frac{1}{a} v_{,x\theta} + \\ & + \tau (A_2 + \frac{\tau}{2} A_1) a \phi_{,xx} + \frac{\tau}{2} (G_2 - \frac{\tau}{2} G_1) \frac{1}{a} \phi_{,\theta\theta} + \frac{\tau}{2} (2B_2 + G_2) \chi_{,x\theta} + \\ & + 2B_1 \frac{1}{a} \omega_{,x} + \tau B_2 \psi_{,x} + \frac{\tau^2}{4} B_1 a \chi_{,x} = -P_x + 2(T_{11} + \frac{\tau}{2} T_{12}),_x \end{aligned} \quad (40)$$

$$\begin{aligned} \delta v: & (2B_1 + G_1) \frac{1}{a} u_{,x\theta} + (G_1 + \frac{\tau}{2} G_2) v_{,xx} + 2 \left[ (1 + \frac{\tau^2}{4}) C_1 - \frac{\tau}{2} C_2 \right] \cdot \\ & \cdot \frac{1}{a^2} v_{,\theta\theta} + \frac{\tau}{2} (2B_2 + G_2) \phi_{,x\theta} + \frac{\tau}{2} (G_2 + \frac{\tau}{2} G_1) a \chi_{,xx} + \tau (C_2 - \frac{\tau}{2} C_1) \cdot \\ & \cdot \frac{1}{a} \chi_{,\theta\theta} + \left[ 2(1 + \frac{\tau^2}{4}) C_1 - \tau C_2 + (1 + \frac{\tau^2}{2}) E \right] \frac{1}{a^2} \omega_{,\theta} - \frac{\tau^2}{2} (C_1 + \frac{1}{6} E) \frac{1}{a} \psi_{,\theta} + \\ & + \frac{\tau^2}{4} (C_1 + \frac{1}{6} E) \chi_{,\theta} - (1 + \frac{\tau^2}{12}) E \frac{1}{a^2} v + E \frac{1}{a} \chi = -P_\theta + \frac{2}{a} T_{21,\theta} \end{aligned} \quad (41)$$

$$\begin{aligned} \delta \omega: & D \omega_{,xx} + (1 + \frac{\tau^2}{12}) E \frac{1}{a^2} \omega_{,\theta\theta} + D \frac{\tau^2}{12} a \psi_{,xx} - E \frac{\tau^2}{12} \frac{1}{a} \psi_{,\theta\theta} \\ & + D \frac{\tau^2}{12} \frac{a^2}{2} \chi_{,xx} + E \frac{\tau^2}{12} \frac{1}{2} \chi_{,\theta\theta} - 2B_1 \frac{1}{a} u_{,x} - \left[ 2(1 + \frac{\tau^2}{4}) C_1 - \right. \\ & - \tau C_2 + (1 + \frac{\tau^2}{12}) E \left. \right] \frac{1}{a^2} v_{,\theta} - (\tau B_2 - D) \phi_{,x} - (\tau C_2 - \frac{\tau^2}{2} C_1 - \\ & - E) \frac{1}{a} \chi_{,\theta} - \left[ 2(1 + \frac{\tau^2}{4}) C_1 - \frac{\tau}{2} C_2 \right] \frac{1}{a^2} \omega - \tau (C_2 - \frac{\tau}{2} C_1) \frac{1}{a} \psi - \\ & - \frac{\tau^2}{4} C_1 \chi = -P_z - \frac{2}{a} T_{21} \end{aligned} \quad (42)$$

$$\begin{aligned} \delta \phi: & \tau (A_2 + \frac{\tau}{2} A_1) u_{,xx} + \frac{\tau}{2} \left[ (1 + \frac{\tau^2}{4}) G_2 - \frac{\tau}{2} G_1 \right] \frac{1}{a^2} u_{,\theta\theta} + \frac{\tau}{2} \cdot \\ & \cdot (2B_2 + G_2) \frac{1}{a} v_{,x\theta} + \frac{\tau^2}{2} (A_1 + \frac{\tau}{2} A_2) a \phi_{,xx} + \end{aligned}$$

$$\begin{aligned}
& + \frac{\tau^2}{4} (G_1 - \frac{\tau}{2} G_2) \frac{1}{a} \phi_{,00} + \frac{\tau^2}{4} (2B_1 + G_1) \chi_{,x0} + (\tau B_2 - D) \frac{1}{a} \omega_{,x} + \frac{\tau^2}{2} \cdot \\
& \cdot (B_1 - \frac{1}{6} D) \psi_{,x} + \frac{\tau^2}{4} (\frac{\tau}{2} B_2 - \frac{1}{6} D) a \chi_{,x} - D \frac{1}{a} \phi = - \frac{1}{a} M_x + \\
& + \tau (T_{12} + \frac{\tau}{2} T_{11}),_x
\end{aligned} \tag{43}$$

$$\begin{aligned}
\delta_x: & \frac{\tau}{2} (2B_2 + G_2) \frac{1}{a} u_{,x0} + \frac{\tau}{2} (G_2 + \frac{\tau}{2} G_1) v_{,xx} + \tau [(1 + \frac{\tau^2}{4}) C_2 - \frac{\tau}{2} C_1] \cdot \\
& \cdot \frac{1}{a^2} v_{,00} + \frac{\tau^2}{4} (2B_1 + G_1) \phi_{,x0} + \frac{\tau^2}{4} (G_1 + \frac{\tau}{2} G_2) a \chi_{,xx} + \frac{\tau^2}{2} \cdot \\
& \cdot (C_1 - \frac{\tau}{2} C_2) \frac{1}{a} \chi_{,00} + [\tau (1 + \frac{\tau^2}{4}) C_2 - \frac{\tau^2}{2} C_1 - (1 + \frac{\tau^2}{12}) E] \frac{1}{a^2} \omega_{,0} + \\
& + \frac{\tau^2}{2} (C_1 - \frac{\tau}{2} C_2 + \frac{1}{6} E) \frac{1}{a} \psi_{,0} + \frac{\tau^2}{4} (\frac{\tau}{2} C_2 - \frac{1}{6} E) \chi_{,0} + \\
& + (1 + \frac{\tau^2}{12}) E \frac{1}{a^2} v - E \frac{1}{a} \chi = - \frac{1}{a} M_\theta + \frac{1}{a} \tau T_{22,0}
\end{aligned} \tag{44}$$

$$\begin{aligned}
\delta_\psi: & D \frac{\tau^2}{12} \omega_{,xx} - E \frac{\tau^2}{12} \frac{1}{a^2} \omega_{,00} + D \frac{\tau^2}{12} a \psi_{,xx} + E \frac{\tau^2}{12} \frac{1}{a} \psi_{,00} - \\
& - \tau B_2 \frac{1}{a} u_{,x} - \tau [(1 + \frac{\tau^2}{4}) C_2 - \frac{\tau}{2} C_1 - \frac{\tau}{12} E] \frac{1}{a^2} v_{,0} - \frac{\tau^2}{2} (B_1 - \frac{1}{6} D) \cdot \\
& \cdot \phi_{,x} - \frac{\tau^2}{2} (C_1 - \frac{\tau}{2} C_2 + \frac{1}{6} E) \frac{1}{a} \chi_{,0} - \tau [(1 + \frac{\tau^2}{4}) C_2 - \frac{\tau}{2} C_1] \frac{1}{a^2} \omega - \\
& - (\frac{\tau^2}{2} C_1 - \frac{\tau^3}{4} C_2 + 2F) \frac{1}{a} \psi - \frac{\tau^2}{4} (\frac{\tau}{2} C_2 + \frac{2}{3} F) \chi = - \frac{1}{a} M_z - \\
& - \frac{1}{a} \tau T_{22} - \frac{2}{a} F (T^{(1)} + \frac{\tau}{4} T^{(2)})
\end{aligned} \tag{45}$$

$$\begin{aligned}
\delta_\chi: & D \frac{\tau^2}{12} \frac{1}{2} \omega_{,xx} + E \frac{\tau^2}{12} \frac{1}{2a^2} \omega_{,00} - B_1 \frac{\tau^2}{4} \frac{1}{a} u_{,x} - \frac{\tau^2}{4} [(1 + \frac{\tau^2}{4}) C_1 - \\
& - \frac{\tau}{2} C_2 + \frac{1}{6} E] \frac{1}{a^2} v_{,0} - \frac{\tau^2}{4} (\frac{\tau}{2} B_2 - \frac{1}{6} D) \phi_{,x} - \frac{\tau^2}{4} (\frac{\tau}{2} C_2 - \frac{\tau^2}{4} C_1 - \frac{1}{6} E) \cdot \\
& \cdot \frac{1}{a} \chi_{,0} - \frac{\tau^2}{4} [(1 + \frac{\tau^2}{4}) C_1 - \frac{\tau}{2} C_2] \frac{1}{a^2} \omega - \frac{\tau^2}{4} (\frac{\tau}{2} C_2 - \\
& - \frac{\tau^2}{4} C_1 + \frac{2}{3} F) \frac{1}{a} \psi - \frac{\tau^2}{8} (\frac{\tau^2}{4} C_1 + \frac{4}{3} F) \chi = - \frac{\tau^3}{8} P_z - \frac{1}{a} \frac{\tau^2}{4} T_{21} - \\
& - \frac{1}{a} \frac{\tau}{2} F (T^{(2)} + \frac{\tau}{3} T^{(3)})
\end{aligned} \tag{46}$$

These equations comprise a set of Flügge-type equations for an orthotropic sandwich cylinders with a weak core, in the presence of temperature gradients. The displacement functions  $u, v, w, \phi, \psi, \chi, \kappa$  appear as unknowns. Further specilizations appear in Section 3.

## 2.4) The Natural Boundary Conditions

### 2.4.1) The stress boundary conditions

The expression for the work  $W_2$  of the edge tractions in the variational integral is a contour integral:

$$W_2 = \int (1 + \frac{z}{a}) dz \oint \bar{T}_i u_i ds. \quad (47)$$

Introducing the components of the traction and the expression for the displacements (1) the work  $W_2$  becomes:

$$\begin{aligned} W_2 = & \int_{-\frac{h}{2}}^{\frac{h}{2}} (1 + \frac{z}{a}) dz \oint [(\tau_{11} n_x + \tau_{21} n_\theta + \tau_{31} n_z)(u + z \phi) + (\tau_{12} n_x \\ & + \tau_{22} n_\theta + \tau_{32} n_z)(v + z \chi)] ds = \int_0^\alpha \left\{ [h' (\tau_{11}' n_x + \tau_{21}' n_\theta) \cdot \right. \\ & \cdot (1 + \frac{z}{a}) + h \tau_{31}^m n_z + h'' (\tau_{11}'' n_x + \tau_{21}'' n_\theta) (1 - \frac{z}{a})] u + \\ & + \frac{h}{2} [h' (\tau_{11}' n_x + \tau_{21}' n_\theta) (1 + \frac{z}{a}) - h'' (\tau_{11}'' n_x + \tau_{21}'' n_\theta) \cdot \\ & \cdot (1 - \frac{z}{a})] \phi \} a d\theta + \int_0^L \left\{ [h' (\tau_{12}' n_x + \tau_{22}' n_\theta) + h \tau_{32}^m n_z + \right. \\ & + h'' (\tau_{12}'' n_x + \tau_{22}'' n_\theta)] v + \frac{h}{2} [h' (\tau_{12}' n_x + \tau_{22}' n_\theta) - \\ & \left. - h'' (\tau_{12}'' n_x + \tau_{22}'' n_\theta)] \chi \right\} dx \end{aligned} \quad (48)$$

We now define stress resultants in the usual manner:

$$\begin{aligned}
 N_{xx} &= (1 + \frac{\epsilon}{2}) \tau_{11}' h' + (1 - \frac{\epsilon}{2}) \tau_{11}'' h'' , \\
 N_{x\theta} &= (1 + \frac{\epsilon}{2}) \tau_{12}' h' + (1 - \frac{\epsilon}{2}) \tau_{12}'' h'' , \\
 N_{\theta x} &= \tau_{21}' h' + \tau_{21}'' h'' , \\
 N_{\theta\theta} &= \tau_{22}' h' + \tau_{22}'' h'' , \\
 Q_x &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{31} (1 + \frac{\epsilon}{2}) dz = h \tau_{31}^m , \\
 Q_\theta &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{32} dz = h \tau_{32}^m , \\
 M_{xx} &= \frac{h}{2} \left[ (1 + \frac{\epsilon}{2}) \tau_{11}' h' - (1 - \frac{\epsilon}{2}) \tau_{11}'' h'' \right] , \\
 M_{x\theta} &= \frac{h}{2} \left[ (1 + \frac{\epsilon}{2}) \tau_{12}' h' - (1 - \frac{\epsilon}{2}) \tau_{12}'' h'' \right] , \\
 M_{\theta x} &= \frac{h}{2} (\tau_{21}' h' - \tau_{21}'' h'' ) , \\
 M_{\theta\theta} &= \frac{h}{2} (\tau_{22}' h' - \tau_{22}'' h'' ) ,
 \end{aligned} \tag{49}$$

and arrive at an expression for  $W_2$ :

$$\begin{aligned}
 W_2 &= \int_0^\alpha \left[ (N_{xx} n_x + N_{x\theta} n_\theta + Q_x n_z) u + (M_{xx} n_x + M_{x\theta} n_\theta) \cdot \right. \\
 &\quad \left. \cdot \phi \right] \alpha d\theta + \int_0^l \left[ (N_{\theta x} n_x + N_{\theta\theta} n_\theta + Q_\theta n_z) v + \right. \\
 &\quad \left. + (M_{\theta x} n_x + M_{\theta\theta} n_\theta) \chi \right] dx .
 \end{aligned} \tag{50}$$

Performing the variation with respect to the displacements,  $\delta W_2 = 0$  yields the natural stress boundary conditions:

$$N_{xx} n_x + N_{x\theta} n_\theta + Q_x n_z = \bar{S}_x ,$$

$$N_{\theta x} n_x + N_{\theta\theta} n_\theta + Q_\theta n_z = \bar{S}_\theta ,$$

$$M_{xx} n_x + M_{x\theta} n_\theta = \bar{R}_x , \quad (51)$$

$$M_{\theta x} n_x + M_{\theta\theta} n_\theta = \bar{R}_\theta .$$

these quantities must be prescribed at  $x=\text{const.}$ ,  $\theta=\text{const.}$

#### 2.4.2) The displacement boundary conditions

Displacement boundary conditions are obtained by variation of the total external work on the shell with respect to the stress resultants. The total work is  $W_1 + W_2$  and follows from (36) and (48):

$$\begin{aligned} W = & \int_0^a \int_0^l [p_x u + m_x \phi + p_\theta v + m_\theta \chi + p_z (\omega + \frac{h^2}{8} \chi) + \\ & + m_z \psi] a dx d\theta + \int_0^a [(N_{xx} n_x + N_{x\theta} n_\theta + Q_x n_z) u + \\ & + (M_{xx} n_x + M_{x\theta} n_\theta) \phi] a d\theta + \int_0^l [(N_{\theta x} n_x + \\ & + N_{\theta\theta} n_\theta + Q_\theta n_z) v + (M_{\theta x} n_x + M_{\theta\theta} n_\theta) \chi] dx . \end{aligned} \quad (52)$$

After variation we have the displacement boundary conditions:

$$u = \bar{u} , \quad \phi = \bar{\phi} , \quad \omega + \frac{h^2}{8} \chi = \bar{\omega} , \quad (53)$$

$$v = \bar{v} , \quad \chi = \bar{\chi} , \quad \psi = \bar{\psi} .$$

(The bars indicate that the quantities must be prescribed in this manner.)



## 2.5) The Stress-Displacement Relations

We substitute the strain-displacement relations (9), (12), and (25) into the stress-strain relations (12) and (23), obtaining:

$$\tau_{11}' = \frac{E'}{1-\nu_1'\nu_2'} \left[ (u_{,x} + \frac{h}{2} \phi_{,x}) + \frac{1}{a'} \nu_2' (v_{,\theta} + \frac{h}{2} \chi_{,\theta} + \frac{h}{2} \psi + \frac{h^2}{8} \chi) - (\alpha_1' + \nu_2' \alpha_2') T' \right] , \quad (54)$$

$$\tau_{12}' = \tau_{21}' = G' \left[ v_{,x} + \frac{h}{2} \chi_{,x} + \frac{1}{a'} (u_{,\theta} + \frac{h}{2} \phi_{,\theta}) \right] , \quad (55)$$

$$\tau_{12}' = \frac{E'}{1-\nu_1'\nu_2'} \frac{\nu_2'}{\nu_1'} \left[ \frac{1}{a'} (v_{,\theta} + \frac{h}{2} \chi_{,\theta} + \omega + \frac{h}{2} \phi + \frac{h^2}{8} \chi) + \nu_1' \cdot (u_{,x} + \frac{h}{2} \phi_{,x}) - (\alpha_2' + \nu_1' \alpha_1') T' \right] , \quad (56)$$

$$\tau_{11}'' = \frac{E''}{1-\nu_1''\nu_2''} \left[ (u_{,x} - \frac{h}{2} \phi_{,x}) + \frac{1}{a''} \nu_2'' (v_{,\theta} - \frac{h}{2} \chi_{,\theta} + \omega - \frac{h}{2} \psi + \frac{h^2}{8} \chi) - (\alpha_1'' + \nu_2'' \alpha_2'') T'' \right] , \quad (57)$$

$$\tau_{12}'' = \tau_{21}'' = G'' \left[ v_{,x} - \frac{h}{2} \chi_{,x} + \frac{1}{a''} (u_{,\theta} - \frac{h}{2} \phi_{,\theta}) \right] , \quad (58)$$

$$\tau_{22}'' = \frac{E''}{1-\nu_1''\nu_2''} \frac{\nu_2''}{\nu_1''} \left[ \frac{1}{a''} (v_{,\theta} - \frac{h}{2} \chi_{,\theta} + \omega - \frac{h}{2} \psi + \frac{h^2}{8} \chi) + \nu_1'' (u_{,x} - \frac{h}{2} \phi_{,x}) - (\alpha_2'' + \nu_1'' \alpha_1'') T'' \right] , \quad (59)$$

$$\tau_{31} = G_x (\phi + \omega_{,x} + z \psi_{,x} + \frac{z^2}{2} \chi_{,x}) , \quad (60)$$

$$\tau_{32} = G_\theta \left[ \chi + \frac{1}{a+z} (\omega_{,\theta} + z \psi_{,\theta} + \frac{z^2}{2} \chi_{,\theta} - v - z \chi) \right] , \quad (61)$$

$$\tau_{33} = E_z (\psi + z \chi - \alpha_c T) . \quad (62)$$

## 2.6) The Stress Resultant Displacement Relations

Inserting the stress-displacement relations (54)-(62) in the definitions for the stress resultant expressions (49); we obtain:

$$N_{xx} = \frac{E'h'}{1-\nu_1'\nu_2'} \left\{ [A_1 u_{,x} + B_1 \frac{1}{a} (v_{,0} + \omega)] + \frac{\varepsilon}{2} [A_2 (u_{,x} + a \phi_{,x}) + B_2 (\chi_{,0} + \psi)] + \frac{\varepsilon^2}{4} (A_1 a \phi_{,x} + B_1 \frac{a}{2} \chi) - (T_{11} + \frac{\varepsilon}{2} T_{12}) \right\}, \quad (63)$$

$$N_{x0} = \frac{E'h'}{1-\nu_1'\nu_2'} \left[ \frac{G_1}{2} (v_{,x} + \frac{1}{a} u_{,0}) + \frac{\varepsilon}{2} \frac{G_2}{2} (v_{,x} + \phi_{,0} + a \chi_{,x}) + \frac{\varepsilon^2}{4} \frac{G_1}{2} a \chi_{,x} \right], \quad (64)$$

$$N_{0x} = \frac{E'h'}{1-\nu_1'\nu_2'} \left[ \frac{G_1}{2} (v_{,x} + \frac{1}{a} u_{,0}) + \frac{\varepsilon}{2} \frac{G_2}{2} (-\frac{1}{a} u_{,0} + \phi_{,0} + a \chi_{,x}) + \frac{\varepsilon^2}{4} \cdot \frac{G_1}{2} (\frac{1}{a} u_{,0} - \phi_{,0}) \right], \quad (65)$$

$$N_{00} = \frac{E'h'}{1-\nu_1'\nu_2'} \left\{ [B_1 u_{,x} + C_1 \frac{1}{a} (v_{,0} + \omega)] + \frac{\varepsilon}{2} [B_2 a \phi_{,x} + C_2 (\chi_{,0} + \psi + \phi - \frac{1}{a} v_{,0} - \frac{1}{a} \omega)] + \frac{\varepsilon^2}{4} C_1 (\frac{a}{2} \chi - \chi_{,0} - \psi + \frac{v_{,0} + \omega}{a}) - T_{21} \right\}, \quad (66)$$

$$M_{xx} = \frac{E'h'h}{2(1-\nu_1'\nu_2')} \left\{ [A_2 u_{,x} + B_2 \frac{1}{a} (v_{,0} + \omega)] + \frac{\varepsilon}{2} [A_1 (u_{,x} + a \phi_{,x}) + B_1 (\chi_{,0} + \psi)] + \frac{\varepsilon^2}{4} (A_2 a \phi_{,x} + B_2 \frac{a}{2} \chi) - (T_{12} + \frac{\varepsilon}{2} T_{11}) \right\}, \quad (67)$$

$$M_{x0} = \frac{E'h'h}{2(1-\nu_1'\nu_2')} \left[ \frac{G_2}{2} (v_{,x} + \frac{1}{a} u_{,0}) + \frac{\varepsilon}{2} \frac{G_1}{2} (v_{,x} + \phi_{,0} + a \chi_{,x}) + \frac{\varepsilon^2}{4} \frac{G_2}{2} a \chi_{,x} \right], \quad (68)$$

$$M_{0x} = \frac{E'h'h}{2(1-\nu_1'\nu_2')} \left[ \frac{G_2}{2} (v_{,x} + \frac{1}{a} u_{,0}) + \frac{\varepsilon}{2} \frac{G_1}{2} (-\frac{1}{a} u_{,0} + \phi_{,0} + a \chi_{,x}) + \frac{\varepsilon^2}{4} \frac{G_2}{2} (\frac{1}{a} u_{,0} - \phi_{,0}) \right], \quad (69)$$

$$M_{\theta\theta} = \frac{E' h^3}{2(1-\nu_1'\nu_2')} \left\{ [B_2 u_{,x} + C_2 \frac{1}{a} (v_{,0} + \omega)] + \frac{\tau}{2} [B_1 a \phi_{,x} + C_1 (\chi_{,0} + \psi - \frac{1}{a} v_{,0} - \frac{1}{a} \omega)] + \frac{\tau^2}{4} C_2 (\frac{a}{2} \chi - \chi_{,0} - \psi + \frac{v_{,0} + \omega}{a}) - T_{22} \right\}, \quad (70)$$

$$Q_x = G_x h \left[ (\phi + \omega_{,x}) + \frac{\tau^2}{12} (a \psi_{,x} + \frac{a^2}{2} \chi_{,x}) \right], \quad (71)$$

$$Q_\theta = G_\theta h \left[ (\chi - \frac{v - \omega_{,0}}{a}) + \frac{\tau^2}{12} (-\psi_{,0} + \frac{a}{2} \chi_{,0} + \frac{\omega_{,0} - v}{a}) \right], \quad (72)$$

$$N_z^{(1)} = \int \tau_{33} dz = E_z h (\psi - T^{(1)}),$$

$$N_z^{(2)} = \frac{4}{h} \int \tau_{33} z dz = E_z h \left( \frac{\tau}{3} a \chi - T^{(2)} \right), \quad (73)$$

$$N_z^{(3)} = \frac{12}{h^2} \int \tau_{33} z^2 dz = E_z h (\psi - T^{(3)}).$$

Equations (63)-(73) constitute the set of stress variables for the shell.

## 2.7) The Equilibrium Equations

The equilibrium conditions can be found by variation of the complementary energy with respect to the displacements. For linear materials the complementary energy is equal to the strain energy expressed in terms of stress resultants times displacements. The stress-resultant displacement relations are introduced as Lagrange multipliers. The procedure follows that described in [28].

2.7.1) The contribution of the facings to the equilibrium equations

$$\begin{aligned}
 U_f = \frac{1}{2} \iint & \left[ \frac{h'}{2} \left( 1 + \frac{\tau}{2} \right) (\epsilon_{11}' \tau_{11}' + 2\epsilon_{12}' \tau_{12}' + \epsilon_{22}' \tau_{22}') + \frac{h''}{2} \left( 1 - \frac{\tau}{2} \right) (\epsilon_{11}'' \tau_{11}'' + 2\epsilon_{12}'' \tau_{12}'' + \right. \\
 & \left. + \epsilon_{22}'' \tau_{22}'') \right] a dx d\theta = \frac{1}{2} \iint \left[ \frac{h'}{2} \left( 1 + \frac{\tau}{2} \right) \left\{ (u_{,x} + \frac{h}{2} \phi_{,x}) \tau_{11}' + \frac{1}{2} [v_{,x} + \frac{h}{2} \chi_{,x} + \right. \right. \\
 & \left. \left. + \frac{1}{a} (u_{,\theta} + \frac{h}{2} \phi_{,\theta}) \right] (\tau_{12}' + \tau_{21}') + \frac{1}{a} (v_{,\theta} + \frac{h}{2} \chi_{,\theta} + \omega + \frac{h}{2} \psi + \frac{h^2}{8} \chi) \tau_{22}' \right\} + \\
 & \left. + \frac{h''}{2} \left( 1 - \frac{\tau}{2} \right) \left\{ (u_{,x} - \frac{h}{2} \phi_{,x}) \tau_{11}'' + \frac{1}{2} [v_{,x} - \frac{h}{2} \chi_{,x} + \frac{1}{a} (u_{,\theta} - \frac{h}{2} \phi_{,\theta}) \right] \right. \\
 & \left. \cdot (\tau_{12}'' + \tau_{21}'') + \frac{1}{a} (v_{,\theta} - \frac{h}{2} \chi_{,\theta} + \omega - \frac{h}{2} \psi + \frac{h^2}{8} \chi) \tau_{22}'' \right\} \right] a dx d\theta \quad (74)
 \end{aligned}$$

We now express (74) in terms of stress resultants times displacements using the relations (63)-(70). The sub-bars of the Lagrange multipliers indicate that the stress resultants are to be expressed in terms of displacements and that the temperature terms must be doubled since they are not multiplied by 1/2 in the expression for the strain energy in (8). Thus, the strain energy plus the Lagrange multipliers reads:

$$\begin{aligned}
 U_f = \frac{1}{2} \iiint & \left[ N_{xx} u_{,x} + M_{xx} \phi_{,x} + N_{x\theta} v_{,x} + M_{x\theta} \chi_{,x} + \right. \\
 & \left. + \frac{1}{a} N_{\theta x} u_{,\theta} + \frac{1}{a} M_{\theta x} \phi_{,\theta} + \frac{1}{a} N_{\theta\theta} (v_{,\theta} + \omega + \right. \\
 & \left. + \frac{\tau^2}{8} a^2 \chi) + \frac{1}{a} M_{\theta\theta} (\chi_{,\theta} + \psi) \right] a dx d\theta + \quad (75) \\
 & \mu_1 (\underline{N_{xx}} - N_{xx}) + \mu_2 (\underline{N_{x\theta}} - N_{x\theta}) + \mu_3 (\underline{N_{\theta x}} - \\
 & - N_{\theta x}) + \mu_4 (\underline{N_{\theta\theta}} - N_{\theta\theta}) + \mu_5 (\underline{M_{xx}} - M_{xx}) \\
 & + \mu_6 (\underline{M_{x\theta}} - M_{x\theta}) + \mu_7 (\underline{M_{\theta x}} - M_{\theta x}) + \\
 & + \mu_8 (\underline{M_{\theta\theta}} - M_{\theta\theta}) \left. \right\} .
 \end{aligned}$$

Performing the variation and allowing the Lagrange multipliers to assume the values

$$\begin{aligned} \mu_1 = u_{,x} \quad , \quad \mu_3 = u_{,\theta} \quad , \quad \mu_5 = \phi_{,x} \quad , \quad \mu_7 = \phi_{,\theta} \quad , \\ \mu_2 = v_{,x} \quad , \quad \mu_4 = v_{,\theta} + \omega + \frac{\tau^2}{8} a^2 x \quad , \quad \mu_6 = \kappa_{,x} \quad , \quad \mu_8 = \kappa_{,\theta} + \psi \quad , \end{aligned} \quad (76)$$

we obtain the seven contributions to the equilibrium conditions:

$$\delta u: -N_{xx,x} - \frac{1}{a} N_{\theta x,\theta} \quad , \quad (77)$$

$$\delta v: -N_{x\theta,x} - \frac{1}{a} N_{\theta\theta,\theta} \quad , \quad (78)$$

$$\delta \omega: \frac{1}{a} N_{\theta\theta} \quad , \quad (79)$$

$$\delta \phi: -M_{xx,x} - \frac{1}{a} M_{\theta x,\theta} \quad , \quad (80)$$

$$\delta \kappa: -M_{x\theta,x} - \frac{1}{a} M_{\theta\theta,\theta} \quad , \quad (81)$$

$$\delta \psi: \frac{1}{a} M_{\theta\theta} \quad , \quad (82)$$

$$\delta \chi: \frac{\tau^2}{8} N_{\theta\theta} \quad . \quad (83)$$

### 2.7.2) The contribution of the core

In view of the weak core assumption, the first equilibrium equation for the core requires:

$$\frac{\partial}{\partial z} \left[ \left(1 + \frac{z}{a}\right) \tau_{13} \right] = 0, \quad (84)$$

and recalling the definition for the shear resultant we find:

$$\tau_{13} = \frac{Q_x}{h} \frac{1}{1 + \frac{z}{a}}. \quad (85)$$

Similarly, the second equilibrium equation for the core is

$$\frac{\partial}{\partial z} \left[ \left(1 + \frac{z}{a}\right)^2 \tau_{23} \right] = 0. \quad (86)$$

This enables us to write the shear stress in terms of the shear resultant:

$$\tau_{23} = \frac{Q_\theta}{h} \frac{1 - \frac{z^2}{4}}{\left(1 + \frac{z}{a}\right)^2}, \quad (87)$$

(see [20]). The entire complementary energy in the core is in accordance with (24)

$$\begin{aligned} U_c = \frac{1}{2} \iiint \left\{ \tau_{13} \left( \phi + \omega_{,x} + z \psi_{,x} + \frac{z^2}{2} \chi_{,x} \right) + \tau_{23} \left[ \chi + \right. \right. \\ \left. \left. + \frac{1}{a+z} \left( \omega_{,\theta} - v + z \psi_{,\theta} - z \chi + \frac{z^2}{2} \chi_{,\theta} \right) \right] + \tau_{33} \left( \psi + \right. \right. \\ \left. \left. + z \chi \right) \right\} \left(1 + \frac{z}{a}\right) a dx d\theta dz = \frac{1}{2} \iiint \left\{ \frac{Q_x}{h} \left( \phi + \omega_{,x} + \right. \right. \\ \left. \left. + z \psi_{,x} + \frac{z^2}{2} \chi_{,x} \right) + \frac{Q_\theta}{h} \left(1 - \frac{z^2}{4}\right) \left[ \chi \left(1 - 2 \frac{z}{a} + 3 \frac{z^2}{a^2}\right) + \right. \right. \\ \left. \left. + \frac{1}{a} \left( \omega_{,\theta} - v + z \psi_{,\theta} - z \chi + \frac{z^2}{2} \chi_{,\theta} \right) \left(1 - \frac{z}{a}\right) \right] + \tau_{33} \left[ \psi + \right. \right. \\ \left. \left. + \left( \frac{\psi}{a} + \chi \right) z + \frac{\chi}{a} z^2 \right] \right\} a dx d\theta dz. \end{aligned} \quad (88)$$

As before, we integrate over the thickness of the shell and include the

Lagrange multipliers:

$$\begin{aligned}
 U_c = \frac{1}{2} \iint \left\{ Q_x \left[ (\phi + \omega_{,x}) + \frac{\tau^2}{12} \frac{a^2}{2} \chi_{,x} \right] + Q_\theta \left( 1 - \frac{\tau^2}{4} \right) \left[ \chi + \frac{1}{a} (\omega_{,\theta} - \right. \right. \\
 \left. \left. - v) \left( 1 + \frac{\tau^2}{12} \right) - (\psi_{,\theta} - \chi) \frac{\tau^2}{12} + \frac{a}{2} \chi_{,\theta} \frac{\tau^2}{12} \right] + N_z^{(1)} \psi + \frac{\tau}{4} N_z^{(2)} \cdot \right. \\
 \left. \cdot (\psi + a\chi) + \frac{\tau^2}{12} N_z^{(3)} a\chi \right\} a dx d\theta + \frac{1}{2} \left[ \mu_9 (Q_x - Q_x) + \mu_{10} (Q_\theta - Q_\theta) + \right. \\
 \left. + \mu_{11} (N_z^{(1)} - N_z^{(1)}) + \mu_{12} (N_z^{(2)} - N_z^{(2)}) + \mu_{13} (N_z^{(3)} - N_z^{(3)}) \right]. \quad (89)
 \end{aligned}$$

Variation with respect to the displacements gives the values of the Lagrange multipliers

$$\begin{aligned}
 \mu_9 = \phi + \omega_{,x} \quad , \quad \mu_{11} = \psi \quad , \\
 \mu_{10} = \chi + \frac{1}{a} (\omega_{,\theta} - v) \quad , \quad \mu_{12} = \frac{\tau}{4} (\psi + a\chi) \quad , \quad \mu_{13} = \frac{\tau^2}{12} a\chi \quad , \quad (90)
 \end{aligned}$$

and the contribution of the core to the equilibrium conditions:

$$\delta u: \equiv 0 \quad , \quad (91)$$

$$\delta v: -\frac{1}{a} Q_\theta \quad , \quad (92)$$

$$\delta \omega: -Q_{x,x} - \frac{1}{a} Q_{\theta,\theta} \quad , \quad (93)$$

$$\delta \phi: Q_x \quad , \quad (94)$$

$$\delta \chi: Q_\theta, \quad (95)$$

$$\delta \psi: -\frac{\tau^2}{24} Q_{x,x} + \frac{\tau^2}{12} \frac{1}{a} Q_{\theta,\theta} + N_z^{(4)} + \frac{\tau}{4} N_z^{(2)}, \quad (96)$$

$$\delta \chi: -\frac{\tau^2}{24} Q_{x,x} - \frac{\tau^2}{24} \frac{1}{a} Q_{\theta,\theta} + \frac{\tau}{4} N_z^{(2)} + \frac{\tau^2}{12} N_z^{(3)}. \quad (97)$$

2.7.3) The contribution of the external loads and the resulting equilibrium equations

Following the same procedure as in Section 2.3 we obtain the load terms as in (39):

$$-P_x, -P_\theta, -P_z, -M_x, -M_\theta, -M_z, -\frac{\tau^2}{8} P_z. \quad (98)$$

Summing up the contributions (77)-(83), (91)-(97) and (98) yields the complete equilibrium equations for a cylindrical sandwich shell:

$$N_{xx,x} + \frac{1}{a} N_{\theta x,\theta} + P_x = 0, \quad (99)$$

$$N_{x\theta,x} + \frac{1}{a} N_{\theta\theta,\theta} + \frac{1}{a} Q_\theta + P_\theta = 0, \quad (100)$$

$$Q_{x,x} + \frac{1}{a} Q_{\theta,\theta} - \frac{1}{a} N_{\theta\theta} + P_z = 0, \quad (101)$$



$$M_{xx,x} + \frac{1}{a} M_{\theta x, \theta} - Q_x + m_x = 0, \quad (102)$$

$$M_{x\theta, x} + \frac{1}{a} M_{\theta\theta, \theta} - Q_\theta + m_\theta = 0, \quad (103)$$

$$\frac{\tau^2}{24} a Q_{x,x} - \frac{\tau^2}{12} Q_{\theta, \theta} - \frac{1}{a} M_{\theta\theta} - N_z^{(1)} - \frac{\tau}{4} N_z^{(2)} + m_z = 0, \quad (104)$$

$$\frac{\tau^2}{24} Q_{x,x} + \frac{\tau^2}{24} \frac{1}{a} Q_{\theta, \theta} - \frac{\tau^2}{8} \frac{1}{a} N_{\theta\theta} - \frac{\tau}{4} \frac{1}{a} N_z^{(2)} - \frac{\tau^2}{12} \frac{1}{a} N_z^{(3)} + \frac{\tau^3}{8} P_z = 0. \quad (105)$$

Note that we find seven equilibrium equations in our problem although only five normally would be expected (the sixth one concerning rotation about the z-axis is satisfied identically). Indeed equations (99)-(103) are the five usual equilibrium equations. (104) and (105) result from variation of the strain energy with respect to the linear and quadratic term in the expansion for the transversal deflection  $w$ , and these two equations represent the linear and the quadratic change of the forces in z-direction when proceeding from one facing to the other through the shell. If we assume incompressibility of the shell in the z-direction, then the above system of equations reduces to that in [20] in the case of a small deformation of a homogeneous cylindrical shell. As will be seen in the sequel, an analogy exists between the equations for a class of sandwich shells and those for a homogeneous shell, provided proper definitions of the stress resultants are introduced.

The correctness of the above equations was checked by substituting (63)-(73) in the stress equilibrium equations (99)-(105). The resulting displacement equations of equilibrium agreed with (40)-(46) except in terms of higher order in the last two equations. This is due to the fact that the core equilibrium equation is contained in (104) and (105), giving an equation for the shear resultant, while the other stress resultants are pure definitions. This is of course an inconsistency in the theory that cannot be avoided. Since the differences are only in the higher order terms, no importance is attached to this peculiarity.

In concluding this section it should be noted that a single application of the Hellinger-Reissner variational theorem could have been employed to obtain the stress resultant equations of equilibrium and the stress resultant-displacement equations, although this procedure produces displacement-stress resultant equations and stress resultant equilibrium equations; hence further algebra is required to obtain results presented here.

### 3. Applications of The Theory

The purpose of this section is to illustrate how the equations obtained in Section 2 reduce to well-known results for special cases. In addition, examples are given which illustrate the use of the equations in order to study viscoelastic effects in the core, as well as temperature effects in an elastic sandwich shell.

### 3.1) The Equations for a Homogeneous Cylinder under the Kirchhoff-Love Assumption

For a shell whose material is isotropic and with facings that are similar, the following specializations are introduced. All constants with the index 2 are set equal to zero (Eq. (15)). Shear effects are neglected (all terms containing the core shear modulus are discarded). The material is assumed to be incompressible in the transverse direction (set  $\psi = 0$  and  $\chi = 0$ ). With these simplifications equations (40)-(44) become:

$$2A_1 u_{,xx} + (1 + \frac{\tau^2}{4}) G_1 \frac{1}{a^2} u_{,\theta\theta} + (2B_1 + G_1) \frac{1}{a} v_{,x\theta} + \frac{\tau^2}{2} A_1 a \phi_{,xx} - \frac{\tau^2}{4} G_1 \frac{1}{a} \phi_{,\theta\theta} + 2B_1 \frac{1}{a} \omega_{,x} = -P_x + 2(T_{11} + \frac{\tau}{2}) T_{12,x} \quad , \quad (106)$$

$$(2B_1 + G_1) \frac{1}{a} u_{,\theta x} + G_1 v_{,xx} + 2(1 + \frac{\tau^2}{4}) C_1 \frac{1}{a^2} v_{,\theta\theta} + \frac{\tau^2}{4} G_1 a \kappa_{,xx} - \frac{\tau^2}{2} C_1 \cdot \frac{1}{a} \kappa_{,\theta\theta} + 2C_1 (1 + \frac{\tau^2}{4}) \frac{1}{a^2} \omega_{,\theta} = -P_\theta + \frac{2}{a} T_{21,\theta} \quad , \quad (107)$$

$$-2B_1 \frac{1}{a} u_{,xx} - 2(1 + \frac{\tau^2}{4}) C_1 \frac{1}{a^2} v_{,\theta} + \frac{\tau^2}{2} C_1 \frac{1}{a^2} \chi_{,\theta} - 2(1 + \frac{\tau^2}{4}) \cdot C_1 \frac{1}{a^2} \omega = -P_z - \frac{2}{a} T_{21} \quad , \quad (108)$$

$$\frac{\tau^2}{2} A_1 a \phi_{,xx} + \frac{\tau^2}{4} G_1 \frac{1}{a} \phi_{,\theta\theta} + \frac{\tau^2}{4} (2B_1 + G_1) \kappa_{,x\theta} + \frac{\tau^2}{2} A_1 u_{,xx} - \frac{\tau^2}{4} G_1 \frac{1}{a^2} u_{,\theta\theta} = -\frac{1}{a} M_x + \tau(T_{12} + \frac{\tau}{2} T_{11})_{,x} \quad , \quad (109)$$

$$\frac{\tau^2}{4} G_1 v_{,xx} - \frac{\tau^2}{2} C_1 \frac{1}{a^2} v_{,\theta\theta} + \frac{\tau^2}{4} (2B_1 + G_1) \phi_{,x\theta} + \frac{\tau^2}{4} G_1 a \kappa_{,xx} + \frac{\tau^2}{2} C_1 \frac{1}{a} \kappa_{,\theta\theta} - \frac{\tau^2}{2} C_1 \frac{1}{a^2} \omega_{,\theta} = -\frac{1}{a} M_\theta + \frac{1}{a} \tau T_{23\theta} \quad . \quad (110)$$

Equations (45) and (46) vanish. For isotropic similar facings we find

$$\begin{aligned} A_1 = C_1 = 2, & \quad T_{11} = T_{21} = (1+\nu) \alpha T_m, \\ B_1 = 2\nu, & \quad T_{12} = T_{22} = (1+\nu) \alpha \Delta T, \\ G_1 = 2(1-\nu), & \quad 2B_1 + G_1 = 2(1+\nu). \end{aligned}$$

The Love-Kirchhoff assumption for displacements implies that

$$\begin{aligned} \phi &= -\omega, x, \\ \chi &= -\frac{1}{a} (\omega, \theta - \nu). \end{aligned} \quad (111)$$

In view of (111) the Euler equations (14) also have to be changed. From [18] setting the variation of the integral

$$I = \iint F(u, u, x, u, \theta, u, x x, u, x \theta, u, \theta \theta) a dx d\theta$$

equal to zero is equivalent to

$$F, u - \frac{\partial}{\partial x} F, u, x - \frac{1}{a} \frac{\partial}{\partial \theta} F, u, \theta + \frac{\partial^2}{\partial x^2} F, u, x x + \frac{1}{a} \frac{\partial^2}{\partial x \partial \theta} F, u, x \theta + \frac{1}{a^2} \frac{\partial^2}{\partial \theta^2} F, u, \theta \theta = 0$$

which now replaces (14) and succeeding variations with respect to displacement variables  $\phi$  and  $\chi$ . In our case the Euler equations can be obtained by combination of

(106)

(107) + (110)

(108) + a (109),<sub>x</sub> + (110),<sub>θ</sub>

Multiplying (106)-(110) by  $a^2/4$  we get, after some computation, the equations:

$$a^2 u_{,xx} + \frac{1-\nu}{2} u_{,\theta\theta} + \nu a \omega_{,x} + \frac{1+\nu}{2} a v_{,x\theta} + \frac{\tau^2}{4} \left( \frac{1-\nu}{2} u_{,\theta\theta} - a^3 \omega_{,xxx} + \right. \\ \left. + \frac{1-\nu}{2} a \omega_{,x\theta\theta} \right) = -\frac{a^2}{4} P_x + \frac{a^2}{2} (1+\nu) \alpha \left( T_m + \frac{\tau}{2} \Delta T \right)_{,x} , \quad (112)$$

$$\frac{1+\nu}{2} a u_{,x\theta} + v_{,\theta\theta} + \frac{1-\nu}{2} a^2 v_{,xx} + \omega_{,\theta} + \frac{\tau^2}{4} \left[ \frac{3}{2} (1-\nu) a^2 v_{,xx} - \right. \\ \left. - \frac{3-\nu}{2} a^2 \omega_{,xx\theta} \right] = -\frac{a^2}{4} P_\theta - \frac{a}{4} M_\theta + (1+\nu) \alpha \frac{a}{2} \left( T_m + \frac{\tau}{2} \Delta T \right)_{,\theta} \quad (113)$$

$$\nu a u_{,x} + v_{,\theta} + \omega + \frac{\tau^2}{4} \left( \frac{1-\nu}{2} a u_{,x\theta\theta} - a^3 u_{,xxx} - \frac{3-\nu}{2} a^2 v_{,xx\theta} + \right. \\ \left. + a^4 \omega_{,xxxx} + 2a^2 \omega_{,xx\theta\theta} + \omega_{,\theta\theta\theta\theta} + 2\omega_{,\theta\theta} + \omega \right) = \frac{a^2}{4} P_z + \quad (114) \\ + \frac{a^2}{4} M_{x,x} + \frac{a}{4} M_{\theta,\theta} - (1+\nu) \frac{\alpha}{2} \left[ a^2 \frac{\tau}{2} (\Delta T + \frac{\tau}{2} T_m)_{,xx} + \right. \\ \left. + \frac{\tau}{2} \Delta T_{,\theta\theta} - T_m \right]$$

If we define an extensional stiffness  $D$  and a bending stiffness  $K$

$$D = \frac{2E'h'}{1-\nu^2} , \quad K = \frac{E'h'h^2}{2(1-\nu^2)} ,$$

and if we consider that  $\frac{K}{a^2 D} = k = \frac{\tau^2}{4}$  , and  $\frac{a^2}{4} P_i = \frac{a^2 P_i}{D}$  ,

then we find complete agreement with the homogeneous shell equations in [19],

pg. 471. In addition our equations contain surface couples and allow an

arbitrary temperature distribution. We see again that the equations of an

homogeneous shell and of an isotropic sandwich shell are identical under proper

definitions of the stiffnesses. It is understood that  $\nu$  is the Poisson's Ratio of the facings; therefore the prime is omitted. The stress-resultant displacement relations also agree with those given in [19], pg. 469/70 Eq.

(1195):

$$N_{xx} = \frac{2E'h'}{1-\nu^2} \left[ u_{,x} + \frac{\nu}{a} (v_{,0} + w) - \frac{\varepsilon^2}{4} a \omega_{,xx} + \frac{1+\nu}{2} \alpha (T_m + \frac{\varepsilon}{2} \Delta T) \right] , \quad (115)$$

$$N_{x0} = 2G'h' \left[ (v_{,x} + \frac{1}{a} u_{,0}) - \frac{\varepsilon^2}{4} (\omega_{,x0} - v_{,x}) \right] , \quad (116)$$

$$N_{0x} = 2G'h' \left[ (v_{,x} + \frac{1}{a} u_{,0}) + \frac{\varepsilon^2}{4} (\omega_{,x0} - \frac{1}{a} u_{,0}) \right] , \quad (117)$$

$$N_{00} = \frac{2E'h'}{1-\nu^2} \left[ \frac{1}{a} (v_{,0} + w) + \nu u_{,x} + \frac{\varepsilon^2}{4} (\omega_{,00} + w) + \frac{1+\nu}{2} \alpha T_m \right] , \quad (118)$$

$$M_{xx} = \frac{E'h'h^2}{2(1-\nu^2)} \frac{1}{a} \left[ (u_{,x} - a \omega_{,xx}) - \frac{\nu}{a} (\omega_{,00} - v_{,0}) + \frac{1+\nu}{2} \alpha (\Delta T + \frac{\varepsilon}{2} T_m) \right] , \quad (119)$$

$$M_{x0} = G'h'h^2 \frac{1}{a} (-\omega_{,0x} + v_{,x}) , \quad (120)$$

$$M_{0x} = G'h'h^2 \frac{1}{a} (-\omega_{,0x} + \frac{1}{2} v_{,x} - \frac{1}{2a} u_{,0}) , \quad (121)$$

$$M_{00} = \frac{E'h'h^2}{2(1-\nu^2)} \frac{1}{a} \left[ -\frac{1}{a} (\omega_{,00} + w) - \nu a \omega_{,xx} + \frac{1+\nu}{2} \alpha \Delta T \right] . \quad (122)$$

### 3.2) The Sandwich Plate Equation According to Reissner's Refined Theory

Assuming that the facings are similar and the material is isotropic, we specialize equations (40)-(44) as follows:

$$\begin{aligned} a\theta &= y, & \psi &= 0, & \tau &= 0, \\ 1/a &= 0, & \chi &= 0. \end{aligned}$$

Since we again assume the sandwich structure to be incompressible in the transverse direction, (45) and (46) vanish. Hence

$$2A_1 u_{,xx} + G_1 u_{,yy} + (2B_1 + G_1) v_{,xy} = -P_x + 2T_{11,x} \quad , \quad (123)$$

$$(2B_1 + G_1) u_{,xy} + G_1 v_{,xx} + 2C_1 v_{,yy} = -P_y + 2T_{21,y} \quad , \quad (124)$$

$$D\omega_{,xx} + E\omega_{,yy} + D\phi_{,x} + E\chi_{,y} = -P_z \quad , \quad (125)$$

$$\begin{aligned} hA_1 \phi_{,xx} + \frac{h}{2} G_1 \phi_{,yy} + \frac{h}{2} (2B_1 + G_1) \chi_{,xy} - \frac{2}{h} D\omega_{,x} - \frac{2}{h} D\phi = \\ = -\frac{2}{h} M_x + 2T_{12,x} \quad , \end{aligned} \quad (126)$$

$$\begin{aligned} \frac{h}{2} (2B_1 + G_1) \phi_{,xy} + \frac{h}{2} G_1 \chi_{,xx} + hC_1 \chi_{,yy} - \frac{2}{h} E\omega_{,y} - \frac{2}{h} E\chi = \\ = -\frac{2}{h} M_y + 2T_{22,y} \quad . \end{aligned} \quad (127)$$

We introduce the constants as on pg. 19 and additionally set

$$D = E = 2(1-\nu^2) \frac{G_c h}{E' h'} \quad .$$

As a result we find two equations describing the in-plane behavior of the plate:

$$2 u_{,xx} + (1-\nu) u_{,yy} + (1+\nu) v_{,xy} = -P_x + 2(1+\nu)\alpha T_{m,x} \quad , \quad (128)$$

$$(1+\nu) u_{,xy} + (1-\nu) v_{,xx} + 2 v_{,yy} = -P_y + 2(1+\nu)\alpha T_{m,y} \quad . \quad (129)$$

and after lengthy manipulations the governing differential equation for the transverse deflection:

$$\nabla^4 w = \frac{2(1-\nu^2)}{E' h^3} (P_z + m_{x,x} + m_{y,y} - \frac{E' h^3}{2(1-\nu^2)G_c} \nabla^2 P_z) - \frac{1+\nu}{h} \alpha \nabla^2 \Delta T \quad . \quad (130)$$

This equation agrees with Resinner's linearized Eq. (70) in [26], with Reissner's Eq. (79) in [25], with Cheng's Eq. (33) in [27] and the temperature term agrees with Melan-Parkus' Eqn. VII, 9 in [21]. The inclusion of the surface couples and of the temperature terms is new and makes the equation more general. The expression  $\frac{E' h^3}{2(1-\nu^2)G_c}$  is the ratio of bending stiffness to shear stiffness. By inspection we see that (130) agrees with the classical plate equation in two cases: (1) if the shear stiffness is infinite (implying the Love-Kirchhoff assumption) (2) if the Laplace operator of the external load vanishes. For a circular plate with a central hole under a uniform shear load at the inner edge (meaning  $\nabla^2 P_z = 0$ ) it can be shown that the stress resultants agree with those of the classical theory but the deflections do not, being dependent on the ratio of bending stiffness to



shear stiffness and hole diameter to plate thickness as well. This result is interesting since one might expect the entire solution to agree with classical theory for this case. In [34] Kao investigated a circular sandwich plate under a linearly varying load and obtained similar results but he did not draw any further conclusions:

A set of ten stress-resultant displacement equations belong to (128)-(130) but are omitted as they can be derived from (63)-(72) easily.

### 3.3) The Plane Strain Problem for a Cylindrical Shell under Axisymmetric Load and Temperature Distribution

Consider an orthotropic sandwich cylinder under a state of plane strain in the  $\theta z$ -plane. We set:

$$\begin{aligned} \frac{\partial}{\partial \theta} = 0, \quad u = 0, \quad \phi = 0, \quad P_\alpha = 0, \\ \frac{\partial}{\partial x} = 0, \quad v = 0, \quad \chi = 0, \quad M_\alpha = 0, \end{aligned} \quad \alpha = x, \theta.$$

We arrive at a set of three algebraic equations for the unknown displacement components  $w$ ,  $\psi$ , and  $\chi$ . A solution is found easily by the determinant rule and the stresses follow from the stress displacement relations. From (42), (45), and (46) can be derived: (the other equations vanish identically)

$$\begin{aligned} \left[ 2\left(1 + \frac{\tau^2}{4}\right) C_1 - \tau_2 C_2 \right] \frac{1}{a^2} w + \tau \left( C_2 - \frac{\tau}{2} C_1 \right) \frac{1}{a} \psi + \frac{\tau^2}{4} C_1 \chi = \\ = P_z + \frac{2}{a} T_{21}, \end{aligned} \quad (131)$$

$$\begin{aligned} & \tau \left[ \left(1 + \frac{\tau^2}{4}\right) C_2 - \frac{\tau}{2} C_1 \right] \frac{1}{a^2} \omega + \left[ \frac{\tau^2}{2} (C_1 - \frac{\tau}{2} C_2) + 2F \right] \frac{1}{a} \psi + \frac{\tau^2}{4} \left( \frac{\tau}{2} C_2 + \frac{2}{3} F \right) \chi = \\ & = \frac{1}{a} M_z + \frac{1}{a} \left[ \tau T_{zz} + 2F(T^{(1)} + T^{(2)} \frac{\tau}{4}) \right], \end{aligned} \quad (132)$$

$$\begin{aligned} & \tau \left[ \left(1 + \frac{\tau^2}{4}\right) C_1 - \frac{\tau}{2} C_2 \right] \frac{1}{a^2} \omega + \tau \left[ \frac{\tau}{2} (C_2 - \frac{\tau}{2} C_1) + \frac{2}{3} F \right] \frac{1}{a} \psi + \tau \left( \frac{\tau^2}{8} C_1 + \right. \\ & \left. + \frac{2}{3} F \right) \chi = \frac{\tau^2}{2} P_z + \frac{1}{a} \tau T_{z1} + \frac{2}{a} F (T^{(2)} + \frac{\tau}{3} T^{(3)}). \end{aligned} \quad (133)$$

### 3.4) Axisymmetric Problems for a Cylindrical Sandwich Shell with a Viscoelastic Core

In the interest of algebraic simplicity we assume isotropic similar facings, an isotropic core, and axisymmetric loads (mechanical and thermal). From (40), (42), (43), (45), and (46) we obtain (the other equations vanish):

$$\begin{aligned} 4u_{,xx} + \tau^2 a \phi_{,xx} + 4\nu \frac{1}{a} \omega_{,x} + \frac{\nu}{2} \tau^2 a \chi_{,x} = -P_x + \\ + 2(1+\nu) \alpha (T_m + \frac{\tau}{2} \Delta T)_{,x}, \end{aligned} \quad (134)$$

$$\begin{aligned} D \omega_{,xx} + D \frac{\tau^2}{12} a (\psi_{,xx} + \frac{a}{2} \chi_{,xx}) - 4\nu \frac{1}{a} u_{,x} + D \phi_{,x} - 4 \left(1 + \frac{\tau^2}{4}\right) \frac{1}{a^2} \omega + \\ + \tau^2 \frac{1}{a} \psi - \frac{1}{2} \tau^2 \chi = -P_z - \frac{\tau}{a} (1+\nu) \alpha T_m, \end{aligned} \quad (135)$$

$$\begin{aligned} \tau^2 (u_{,xx} + a \phi_{,xx}) - D \frac{1}{a} (\omega_{,x} + \phi) + \tau^2 \nu \psi_{,x} - \frac{\tau^2}{12} D (\psi_{,x} + \\ + \frac{a}{2} \chi_{,x}) = -\frac{1}{a} M_x + (1+\nu) \alpha \tau (\Delta T + \frac{\tau}{2} T_m)_{,x}, \end{aligned} \quad (136)$$

$$\begin{aligned} \frac{\tau^2}{12} D (\omega_{,xx} + a \psi_{,xx} + \phi_{,x}) - \nu \tau^2 \phi_{,x} + \tau^2 \frac{1}{a^2} \omega - \left( \frac{\tau^2}{2} + 2F \right) \frac{1}{a} \psi - \\ - \frac{\tau^2}{6} F \chi = -\frac{1}{a} M_z - \frac{1}{a} \left[ \tau (1+\nu) \alpha \Delta T + 2F (T^{(1)} + \frac{\tau}{4} T^{(2)}) \right]. \end{aligned} \quad (137)$$

$$\begin{aligned} \frac{\tau^2}{24} D(\omega_{,xy} + \phi_{,x}) - \frac{\tau^2}{2} \nu \frac{1}{a} u_{,x} - \frac{\tau^2}{2} (1 + \frac{\tau^2}{4}) \frac{1}{a^2} \omega + \frac{\tau^2}{4} (\frac{\tau^2}{2} - \frac{2}{3} F) \cdot \\ \cdot (\frac{1}{a} \psi - \chi) = -\frac{\tau^3}{8} P_z - \frac{1}{a} \frac{\tau}{4} \left[ \tau(1+\nu) \alpha T_m + 2F(T^{(1)} + \frac{\tau}{3} T^{(2)}) \right] . \end{aligned} \quad (138)$$

In order to study viscoelastic effects (limited to the core) we have to replace the constitutive equations for the core by other equations that describe the viscoelastic behavior. We utilize the integral constitutive law which is derived from Boltzmann's superposition principle:

$$\begin{aligned} \tau_{13} &= \int_0^t G_c(t-t') \frac{\partial \epsilon_{13}}{\partial t'} dt' , \\ \tau_{23} &= \int_0^t G_c(t-t') \frac{\partial \epsilon_{23}}{\partial t'} dt' , \\ \tau_{33} &= \int_0^t E_z(t-t') \frac{\partial}{\partial t'} (\epsilon_{33} - \alpha_c T) dt' . \end{aligned} \quad (139)$$

We then apply the Laplace transform for the time coordinate. By observing the Laplace transform of a convolution integral we find the Laplace transform of  $Dw_{,xx}$ :

$$\frac{h}{Gh'} \int_0^\infty e^{-pt} dt \int_0^t G_c(t-t') \frac{\partial \omega_{,xx}}{\partial t'} dt' = p D^* \omega^*_{,xx} ,$$

where at rest initial conditions are assumed. Thus all products of core modulus times displacement are convolution integrals. Denoting the Laplace transform of a quantity by an asterisk, as above, the five partial differential

equations of equilibrium in Laplace space are:

$$4(u_{,xx}^* + \frac{\gamma}{2} \omega_{,x}^*) + \tau^2 a (\phi_{,xx}^* + \frac{\gamma}{2} \chi_{,x}^*) = -P_x^* + 2(1+\nu) \alpha (T_{m,x}^* + \frac{\gamma}{2} \Delta T_{,x}^*), \quad (140)$$

$$PD^* [\omega_{,xx}^* + \phi_{,x}^* + \frac{\tau^2}{12} a (\psi_{,xx}^* + \frac{a}{2} \chi_{,xx}^*)] - 4 [\frac{\gamma}{2} u_{,x}^* + (1 + \frac{\tau^2}{4}) \frac{1}{a^2} \omega^*] + \tau^2 (\frac{1}{2} \psi^* - \frac{1}{2} \chi^*) = -P_z^* - \frac{\gamma}{2} (1+\nu) \alpha T_m^*, \quad (141)$$

$$-PD^* [\frac{1}{2} (\omega_{,x}^* + \phi^*) + \frac{\tau^2}{12} (\psi_{,x}^* + \frac{a}{2} \chi_{,x}^*)] + \tau^2 (u_{,xx}^* + a \phi_{,xx}^* + \nu \psi_{,xx}^*) = -\frac{1}{2} M_x^* + (1+\nu) \alpha \tau (\Delta T_{,x}^* + \frac{\gamma}{2} T_{m,x}^*), \quad (142)$$

$$PD^* \frac{\tau^2}{12} (\omega_{,xx}^* + \phi_{,x}^* + a \psi_{,xx}^*) - \tau^2 (\nu \phi_{,x}^* - \frac{1}{a^2} \omega^* + \frac{1}{2a} \psi^*) - 2PF^* \cdot (\frac{1}{2} \psi^* + \frac{\tau^2}{12} \chi^*) = -\frac{1}{2} M_z^* - \frac{1}{2} [\tau(1+\nu) \alpha \Delta T^* + 2PF^* (T^{(1)*} + \frac{\gamma}{4} T^{(2)*})], \quad (143)$$

$$\frac{\tau^2}{24} PD^* (\omega_{,xx}^* + \phi_{,x}^*) - \frac{1}{2} \frac{\tau^2}{2} [\nu u_{,x}^* + (1 + \frac{\tau^2}{4}) \frac{1}{a^2} \omega^* - \frac{\tau^2}{4} (\psi^* - a \chi^*)] - \frac{\tau^2}{6} PF^* \frac{1}{2} (\psi^* - a \chi^*) = -\frac{\tau^2}{8} P_z^* - \frac{1}{2} \frac{\tau^2}{2} [\frac{\gamma}{2} (1+\nu) \alpha T_m^* + PF^* (T^{(2)*} + \frac{\gamma}{3} T^{(3)*})]. \quad (144)$$

In order to obtain equations that can be handled without excessive numerical work we drop (143) and (144), implying incompressibility of the shell in the transverse direction. We then tried to find a single governing differential equation which could be used to study various effects on the cylinder. The following steps were undertaken:

- a) All terms of order  $\tau^2$  were dropped. This meant that all bending terms were eliminated and that only uniform heating or uniformly distributed loads could have been studied.
- b) The Kirchhoff-Love assumption was made:  $\phi = -w_{,x}$ . This meant that all terms containing the core shear stiffness vanished. No viscoelastic effects could have been studied.
- c) The shell was assumed to be restrained in x-direction. It was found that the governing differential equation was not simpler than that one obtained under d), but much less general.
- d) No terms were dropped at all. The governing differential equation was obtained by a procedure to be shown subsequently. The manipulation

$$\begin{aligned}
 & \left\{ (PD^* + 4\gamma)\tau^2 \frac{\partial}{\partial x} - 4\gamma PD^* \int dx \right\} \cdot (142) + \\
 & + \left\{ 4\left[\tau^2\left(1 - \frac{\tau^2}{4}\right) \alpha \frac{\partial^2}{\partial x^2} - PD^* \frac{1}{a}\right] \right\} \cdot (143) + \\
 & + \left\{ (-4)(PD^* + \gamma\tau^2) \frac{\partial}{\partial x} \right\} \cdot (144)
 \end{aligned}$$

yields the governing differential equation for a cylindrical sandwich shell with axisymmetric load and temperature distribution and including viscoelastic shear effects in the core:

$$\begin{aligned} & \rho D^* \tau^2 a^3 \omega_{,xxxx}^* + 2 [\rho D^* \nu - 2(1-\nu^2)] \tau^2 a \omega_{,xx}^* + 4 \rho D^* (1-\nu^2 + \frac{\tau^2}{4}) \cdot \\ & \cdot \frac{1}{a} \omega^* = (-\tau^2 a^3 P_{z,xx} + \rho D^* a P_z) + 2(1+\nu) \alpha [(1-\nu)(\rho D^* T_m^* - \\ & - \tau^2 a^2 T_{m,xx}^*) - \rho D^* \tau (\nu \Delta T^* + a^2 (1 - \frac{\tau^2}{4}) \Delta T_{,xx}^*)] . \end{aligned} \quad (145)$$

In addition there results

$$\begin{aligned} & \tau^2 a^4 \omega_{,xxxx} + 2 [\nu - \frac{2(1-\nu^2)}{D}] \tau^2 a^2 \omega_{,xx} + 4 (1-\nu^2 + \frac{\tau^2}{4}) \omega = \\ & = (-\frac{\tau^2}{D} a^4 P_{z,xx} + a^2 P_z) + 2(1+\nu) \alpha a [(1-\nu)(T_m - \frac{\tau^2}{D} a^2 T_{m,xx}) - \\ & - \frac{\tau}{2} (\nu \Delta T + a^2 (1 - \frac{\tau^2}{4}) \Delta T_{,xx})] . \end{aligned} \quad (146)$$

Equation (146) is the differential equation for an elastic shell with elastic core and can be checked in the following manner:

The equations (112) and (113) are in the case of axisymmetry:

$$a^2 u_{,xx} + \nu a \omega_{,x} - \frac{\tau^2}{4} a^3 \omega_{,xxx} = \frac{a^2}{2} (1+\nu) \alpha (T_m + \frac{\tau}{2} \Delta T)_{,x} , \quad (147)$$

$$\begin{aligned} & \nu a u_{,x} + \omega - \frac{\tau^2}{4} a^3 u_{,xxx} + \frac{\tau^2}{4} a^4 \omega_{,xxxx} + \frac{\tau^2}{4} \omega = \\ & = \frac{a^2}{4} P_z - 2(1+\nu) \alpha [a^2 \frac{\tau}{2} (\Delta T + \frac{\tau}{2} T_m)_{,xx} - T_m] . \end{aligned} \quad (148)$$

By performing the operation

$$\frac{4}{a^2 \tau^2} \left( -\frac{\nu}{a} \int (147) dx + \frac{\tau^2}{4} a \frac{\partial}{\partial x} (148) \right) ,$$

and by letting the shear modulus approach infinity ( $D \rightarrow \infty$ ) in (146) we find two identical equations. The term  $w_{,xx}$  is small compared with  $a^2 w_{,xxxx}$  and  $w/a^2 \tau^2$  and can be neglected. This leads to the well known differential equation for axisymmetric deformation of a cylindrical shell and solutions are of the same type as the solutions for a beam on elastic foundation.

### 3.5) Example: Spatial Distribution of Temperature in an Elastic Sandwich Cylinder (elastic core)

As an example consider an isotropic elastic sandwich shell with temperature independent material properties.

Equation (146) is used to study the effect of heating the shell along a distance  $2a_T$ . The temperature is assumed to be uniform through the thickness and no external loads are considered. The integral transform technique is used to solve the differential equation, in particular the Fourier transform. Regularity requirements are satisfied as  $w$  and all its derivatives vanish for  $x$  approaching infinity. Upon setting:  $P_z = 0$ ,  $T = 0$ ,  $T_m = T'$  we obtain from (146):

$$Da^3 \tau^2 \omega_{,xxxx} + 2 [D\tau - 2(1-\nu^2)] \tau^2 a \omega_{,xx} + 4D(1-\nu^2 + \frac{\tau^2}{4}) \frac{1}{2} \omega = -2(1+\nu) \alpha [(1-\nu)(DT' - \tau^2 a T'_{,xx})] . \quad (149)$$

The Fourier transform of the temperature distribution is: (see Fig. 3)

$$\hat{T} = T' \frac{2 \sin(s a_T)}{s} ,$$

where  $(\hat{\phantom{x}})$  denotes the Fourier transform defined by

$$\hat{\omega}(s) = \int_{-\infty}^{\infty} \omega(x) e^{isx} dx.$$

The expression for the radial deflection in the transform space is:

$$\hat{\omega} = \frac{2(1-\nu^2)\alpha T^1(D - \tau^2 a^2 s^2) 2 \sin(s a_T)}{s \{ D \tau^2 a^3 s^4 - 2[D\nu - 2(1-\nu^2)] \tau^2 a s^2 + 4D(1-\nu^2 + \frac{\tau^2}{4}) \frac{1}{a} \}} \quad (150)$$

By defining a dimensionless coordinate

$$\eta = sa, \quad \lambda = \frac{a_T}{a},$$

we can rewrite (150):

$$\frac{\hat{\omega}}{a^2} = - \frac{4(1-\nu^2)\alpha T^1}{D} \frac{\eta^2 - \frac{D}{\tau^2}}{\eta \{ \eta^4 - 2[\nu - \frac{2(1-\nu^2)}{D}] \eta^2 + [1 + \frac{1-\nu^2}{\tau^2/4}] \}} \quad (151)$$

The denominator that is needed to find the inverse transform is of the form:

$$\eta^4 - B\eta^2 + C = (\eta^2 - \chi^2)(\eta^2 + \delta^2),$$

where

$$B = 2[\nu - \frac{2(1-\nu^2)}{D}], \quad C = 1 + \frac{1-\nu^2}{\tau^2/4},$$

$$\chi^2 = \frac{1}{2}(B + i\sqrt{4C - B^2}), \quad \delta^2 = \frac{1}{2}(B - i\sqrt{4C - B^2}).$$



The deflection is found by taking the inverse transform

$$\begin{aligned} \frac{\omega}{a} &= \frac{4}{2\pi} \int_0^\infty \left( \frac{\hat{\omega}}{a} \right) \cos\left(\frac{x}{a} \eta\right) d\eta = \\ &= - \frac{4(1-\nu^2)\alpha T'}{D\pi} \int_0^\infty \overbrace{\frac{\sin(\lambda\eta) (\eta^2 - \frac{D}{2\tau})}{\eta(\eta^2 - \gamma^2)(\eta^2 + \delta^2)}}^{\hat{I}} \cos\left(\frac{x}{a} \eta\right) d\eta. \end{aligned} \quad (152)$$

The expression  $\hat{I}$  is rewritten as

$$\hat{I} = \frac{1}{\gamma^2 + \delta^2} \left[ \frac{(\gamma^2 - D/2\tau) \sin(\lambda\eta)}{\eta[\eta^2 + (-i\gamma)^2]} + \frac{(\delta^2 + D/2\tau) \sin(\lambda\eta)}{\eta(\eta^2 + \delta^2)} \right]. \quad (153)$$

The inverse transform of  $\hat{I}$  must be found separately for  $x > a_T$  and  $x < a_T$ .

Using [23], pg. 19 and auxiliary quantities

$$\cos \frac{\phi}{2} = \frac{1}{2} \sqrt{\frac{2\sqrt{C} + B}{\sqrt{C}}}, \quad \sin \frac{\phi}{2} = \frac{1}{2} \sqrt{\frac{2\sqrt{C} - B}{\sqrt{C}}}, \quad \phi = \arctan \sqrt{\frac{2\sqrt{C} - B}{2\sqrt{C} + B}},$$

we find

$$\begin{aligned} \frac{2I}{\pi} &= -\frac{D}{2C\tau} \left\{ \exp\left[-\sqrt[4]{C} \left(\frac{x}{a} - \lambda\right) \sin \frac{\phi}{2}\right] \cos\left[\sqrt[4]{C} \left(\frac{x}{a} - \lambda\right) \cos \frac{\phi}{2}\right] - \right. \\ &\quad \left. - \exp\left[-\sqrt[4]{C} \left(\frac{x}{a} + \lambda\right) \sin \frac{\phi}{2}\right] \cos\left[\sqrt[4]{C} \left(\frac{x}{a} + \lambda\right) \cos \frac{\phi}{2}\right] \right\} + \\ &\quad + \frac{BD - 2C\tau^2}{2C\tau^2 \sqrt{4C - B^2}} \left\{ \exp\left[-\sqrt[4]{C} \left(\frac{x}{a} - \lambda\right) \sin \frac{\phi}{2}\right] \sin\left[\sqrt[4]{C} \left(\frac{x}{a} - \lambda\right) \cos \frac{\phi}{2}\right] - \right. \\ &\quad \left. - \exp\left[-\sqrt[4]{C} \left(\frac{x}{a} + \lambda\right) \sin \frac{\phi}{2}\right] \sin\left[\sqrt[4]{C} \left(\frac{x}{a} + \lambda\right) \cos \frac{\phi}{2}\right] \right\}. \end{aligned} \quad (154)$$

The solution for  $x > a_T$  is given by

$$\begin{aligned} \frac{\omega/a}{2\alpha T} = & \frac{1-\gamma^2}{\tau^2 C} \left\{ \frac{1}{2} \left( \exp\left[-\frac{1}{2}\sqrt{2\tau C-B} \left(\frac{x}{a}-\lambda\right)\right] \cos\left[\frac{1}{2}\sqrt{2\tau C+B} \left(\frac{x}{a}-\lambda\right)\right] - \right. \right. \\ & - \exp\left[-\frac{1}{2}\sqrt{2\tau C-B} \left(\frac{x}{a}+\lambda\right)\right] \cos\left[\frac{1}{2}\sqrt{2\tau C+B} \left(\frac{x}{a}+\lambda\right)\right] \Big) + \frac{C\tau^2}{D} - \frac{B}{2} \cdot \\ & \cdot \left( \exp\left[-\frac{1}{2}\sqrt{2\tau C-B} \left(\frac{x}{a}-\lambda\right)\right] \sin\left[\frac{1}{2}\sqrt{2\tau C+B} \left(\frac{x}{a}-\lambda\right)\right] - \right. \\ & \left. \left. - \exp\left[-\frac{1}{2}\sqrt{2\tau C-B} \left(\frac{x}{a}+\lambda\right)\right] \sin\left[\frac{1}{2}\sqrt{2\tau C+B} \left(\frac{x}{a}+\lambda\right)\right] \right) \right\}, \end{aligned} \quad (155)$$

and at the origin: ( $x=0$ )

$$\begin{aligned} \frac{\omega/a}{2\alpha T} = & \frac{1-\gamma^2}{\tau^2 C} \left\{ \frac{1}{2} \left( 1 - \exp\left[-\sqrt{2\tau C-B} \lambda\right] \cos\left[\sqrt{2\tau C+B} \lambda\right] \right) + \right. \\ & \left. + \frac{C\tau^2}{D} - \frac{B}{2} \left( -\exp\left[-\sqrt{2\tau C-B} \lambda\right] \sin\left[\sqrt{2\tau C+B} \lambda\right] \right) \right\}. \end{aligned} \quad (156)$$

It can be noted that the solution damps out for  $x \rightarrow \infty$ . Similarly an inverse transform is found for  $x < a_T$ :

$$\begin{aligned} \frac{2I}{\pi} = & -\frac{D}{C\tau^2} \left\{ 1 - \frac{1}{2} \left( \exp\left[-\sqrt{\tau C} \left(\lambda + \frac{x}{a}\right) \sin \frac{\phi}{2}\right] \cos\left[\sqrt{\tau C} \left(\lambda + \frac{x}{a}\right) \cos \frac{\phi}{2}\right] + \right. \right. \\ & \left. \left. + \exp\left[-\sqrt{\tau C} \left(\lambda - \frac{x}{a}\right) \sin \frac{\phi}{2}\right] \cos\left[\sqrt{\tau C} \left(\lambda - \frac{x}{a}\right) \cos \frac{\phi}{2}\right] \right) \right\} + \\ & + \frac{C\tau^2 - \frac{BD}{2}}{C\tau^2 \sqrt{4C-B^2}} \left\{ \exp\left[-\sqrt{\tau C} \left(\lambda + \frac{x}{a}\right) \sin \frac{\phi}{2}\right] \sin\left[\sqrt{\tau C} \left(\lambda + \frac{x}{a}\right) \cos \frac{\phi}{2}\right] + \right. \\ & \left. + \exp\left[-\sqrt{\tau C} \left(\lambda - \frac{x}{a}\right) \sin \frac{\phi}{2}\right] \sin\left[\sqrt{\tau C} \left(\lambda - \frac{x}{a}\right) \cos \frac{\phi}{2}\right] \right\}, \end{aligned} \quad (157)$$

with the solution

$$\begin{aligned} \frac{\omega/a}{2\alpha T'} = \frac{1-\gamma^2}{c^2 C} \left\{ \left[ 1 - \frac{1}{2} \left( \exp\left[-\frac{1}{2}\sqrt{2\tau C - B} \left(\lambda + \frac{x}{a}\right)\right] \cos\left[\frac{1}{2}\sqrt{2\tau C + B} \left(\lambda + \frac{x}{a}\right)\right] + \right. \right. \right. \\ \left. \left. \left. + \exp\left[-\frac{1}{2}\sqrt{2\tau C - B} \left(\lambda - \frac{x}{a}\right)\right] \cos\left[\frac{1}{2}\sqrt{2\tau C + B} \left(\lambda - \frac{x}{a}\right)\right] \right) \right] - \frac{C\tau^2}{B} - \frac{B}{2} \cdot \right. \\ \left. \cdot \left( \exp\left[-\frac{1}{2}\sqrt{2\tau C - B} \left(\lambda + \frac{x}{a}\right)\right] \sin\left[\frac{1}{2}\sqrt{2\tau C + B} \left(\lambda + \frac{x}{a}\right)\right] + \right. \right. \\ \left. \left. + \exp\left[-\frac{1}{2}\sqrt{2\tau C - B} \left(\lambda - \frac{x}{a}\right)\right] \sin\left[\frac{1}{2}\sqrt{2\tau C + B} \left(\lambda - \frac{x}{a}\right)\right] \right) \right\} \cdot \end{aligned} \quad (158)$$

The value of (158) at the origin,

$$\begin{aligned} \frac{\omega/a}{2\alpha T'} = \frac{1-\gamma^2}{c^2 C} \left[ \frac{1}{2} \left( 1 - \exp\left[-\sqrt{2\tau C - B} \lambda\right] \cos\left[\sqrt{2\tau C + B} \lambda\right] \right) - \right. \\ \left. - \frac{C\tau^2}{B} - \frac{B}{2} \left( \exp\left[-\sqrt{2\tau C - B} \lambda\right] \sin\left[\sqrt{2\tau C + B} \lambda\right] \right) \right], \end{aligned} \quad (159)$$

agrees with (156). For  $\lambda \rightarrow \infty$ , corresponding to uniform heating of the shell, we obtain from (159)

$$\frac{\omega}{2\alpha T'a} = \frac{1-\gamma^2}{c^2 \tau} = \frac{1}{4}. \quad (160)$$

The same solution can be obtained from (145) if we drop all derivatives:

$$\begin{aligned} 4D \left( 1 - \gamma^2 + \frac{\tau^2}{4} \right) \frac{1}{a} \omega = 2(1-\gamma^2) \alpha D T', \\ \omega = \frac{2(1-\gamma^2) \alpha T'a}{4 \left( 1 - \gamma^2 + \frac{\tau^2}{4} \right)} = 2\alpha T'a \frac{1-\gamma^2}{c^2 \tau}. \end{aligned} \quad (161)$$

On the other hand (131) yields a different result since we assumed plane strain. In that case:

$$\omega = \frac{(1+\nu) \alpha T' a}{2 (1 + \frac{\tau^2}{\pi})} \quad (162)$$

The deflections represented by (155) and (158) are plotted under the assumptions:

$$\lambda=1, \quad \tau = 1/50, \quad \nu = 0.3, \quad D=1.$$

Computation of the quantities

$$B = -3.04, \quad \sqrt{2\tau C - B} = 13.70, \quad \frac{1-\nu^2}{\tau^2 C} = 0.249992,$$

$$C = 9101, \quad \sqrt{2\tau C + B} = 13.92, \quad \frac{1-\nu^2}{C\tau^2} \frac{\frac{C\tau^2}{D} - \frac{B}{2}}{\sqrt{4C - B^2}} = 0.002778,$$

leads to the table of the  $w/2a \alpha T'$  values below (see also Fig. 4):

$x/a$	0	0.1	0.3	0.5	0.7	0.9	1.0	1.1	1.2	1.5	2.0
$\frac{w/a}{2\alpha T'}$	0.2498	0.2497	0.2498	0.2546	0.2578	0.2016	0.1250	0.0484	0.0057	0.0038	0.0001

The value of the deflection for  $\lambda \rightarrow \infty$  is given by:

$$\frac{\omega/a}{2\alpha T'} = \frac{1-\nu^2}{C\tau^2} = 1/4. \quad (163)$$

We now consider response due to a temperature at the origin. From (158) we have the expression

$$\frac{\omega/a}{2\alpha T'} = \frac{1-\gamma^2}{\tau^2 C} \left[ (1 - \exp[-\frac{1}{2}\sqrt{2\sqrt{C}-B}\lambda]) \cos[\frac{1}{2}\sqrt{2\sqrt{C}+B}\lambda] - \right. \\ \left. - 2 \frac{\frac{C\tau^2}{D} - \frac{B}{2}}{\sqrt{4C-B^2}} \exp[-\frac{1}{2}\sqrt{2\sqrt{C}-B}\lambda] \sin[\frac{1}{2}\sqrt{2\sqrt{C}+B}\lambda] \right], \quad (164)$$

which vanishes for  $\lambda=0$  clearly because the heated length is set equal to zero. If we wish to examine ring heating of the cylinder we have to perform a limiting process. We assume that the temperature does not increase ad infinitum at the origin; thus

$$T'\lambda = \text{const. } T_0,$$

$$\text{and} \quad \lim_{\lambda \rightarrow 0} T' I(x, \lambda) = \lim_{\lambda \rightarrow 0} \frac{T_0}{\lambda} I(x, \lambda) = T_0 \lim_{\lambda \rightarrow 0} \frac{I(x, \lambda)}{\lambda}.$$

Furthermore, observing that

$$\lim_{\lambda \rightarrow 0} \frac{\text{sh}(-i\gamma\lambda)}{\lambda} = -i\gamma, \quad \lim_{\lambda \rightarrow 0} \frac{\text{sh}(\lambda\delta)}{\lambda} = \delta,$$

we obtain the inverse transform of  $\hat{I}$ :

$$\frac{2I}{\pi} = \frac{-iD}{\tau^2 C^2} \sqrt[4]{C} \exp[-\sqrt[4]{C} \frac{x}{a} \sin \frac{\phi}{2}] 2 \cos[\frac{\phi}{2} - \sqrt[4]{C} \frac{x}{a} \cos \frac{\phi}{2}] + \\ + i\sqrt[4]{C} \exp[-\sqrt[4]{C} \frac{x}{a} \sin \frac{\phi}{2}] 2 \cos[\frac{\phi}{2} + \sqrt[4]{C} \frac{x}{a} \cos \frac{\phi}{2}],$$

and the solution for ring heating of the shell:

$$\frac{\omega/a}{2\alpha T_0} = \frac{2(1-\gamma^2)}{\tau^2 C} \sqrt[4]{C} \exp[-\frac{1}{2}\sqrt{2\sqrt{C}-B} \frac{x}{a}] \left\{ \sqrt{\frac{C}{4C-B^2}} \cdot \right. \\ \left. \cdot \cos[\frac{\phi}{2} - \frac{1}{2}\sqrt{2\sqrt{C}+B} \frac{x}{a}] - \frac{\tau^2 C}{D\sqrt{4C-B^2}} \cos[\frac{\phi}{2} + \frac{1}{2}\sqrt{2\sqrt{C}+B} \frac{x}{a}] \right\}. \quad (165)$$

The deflection at the origin is:

$$\frac{w/a}{2\alpha T_0} = \frac{1-\nu^2}{\tau^2 C} \frac{\sqrt{2} - \frac{\tau^2 C}{B}}{\sqrt{2\sqrt{2}-B}}. \quad (166)$$

Evaluation of (165) leads to the table of values of  $w/2a\alpha T_0$  below (see also Fig. 5):

$\frac{x}{a}$	0	0.1	0.2	0.4
$\frac{w/a}{2\alpha T_0}$	1.674	1.223	0.522	-0.062

### 3.6) Example: Uniform Heating of a Cylindrical Sandwich Shell With a Ringload at the Origin (Viscoelastic Core)

We consider a shell with a viscoelastic core whose temperature dependence is that of a thermorheologically simple material.

The study of temperature effects in viscoelastic materials makes sense only when we take into account the temperature influence on material properties. The following treatment is drawn from references [29] - [33] and a solution found in a manner similar to [17]. We start from (145):

$$Da^3 \tau^2 \omega_{,xxxx} + 2[D\nu - 2(1-\nu^2)] \tau^2 a \omega_{,xx} + 4D(1-\nu^2 + \frac{\tau^2}{\eta}) \cdot \\ \cdot \frac{1}{\alpha} \omega = 2(1-\nu^2) \alpha_0 \theta D + (-\tau^2 a^3 P_{z,xx} + Da P_z). \quad (167)$$

under the assumptions:  $T_m = T = T(t)$  only

$$\Delta T = 0$$

$$\alpha = \alpha(T)$$

$$P_z = P_z(x) \cdot H(t)$$

$\alpha_0 \theta$  is called the pseudo-temperature, defined by

$$\alpha_0 \theta = \int_{T_0}^{T(t)} \alpha(T') dT', \quad \text{where } T_0 = \text{reference temperature.}$$

We now focus our attention on the linear integral operator D:

$$D = \frac{h}{G'h'} \int_{-\infty}^t G_c(t-t'; T) \frac{\partial}{\partial t'} \left\{ \quad \right\} dt'.$$

For thermorheologically simple materials it is possible to introduce a transformation such that  $D(t, T) = D(\xi)$ , where

$$\xi(t) = \int_0^t \phi[T(t')] dt'.$$

$\phi(T)$  is called the shift function and must be adduced experimentally for a given material. Thus the operator D can be transformed to

$$D(\xi) = \frac{h}{G'h'} \int_{-\infty}^{\xi} G_c(\xi - \xi') \frac{\partial}{\partial \xi'} \left\{ \quad \right\} d\xi',$$

and we denote all functions in which  $t$  is replaced by  $\xi$  by

$$\hat{f}(\xi) = f[t = g(\xi)], \quad \text{where } \xi \text{ is called the reduced time.}$$

The solution of (167) can be separated into two parts:

$$\hat{\omega}(x, \xi) = \hat{\omega}_1(x, \xi) + \hat{\omega}_2(\xi). \quad (168)$$

This leads to the equations

$$4D(1-\gamma^2 + \frac{\xi^2}{4}) \frac{1}{\alpha} \hat{\omega}_2 = 2(1-\gamma^2) \alpha_0 \theta D, \quad (169)$$

and

$$\begin{aligned} \tau^2 a^3 D \hat{\omega}_{1,xxxx} + 2[D\gamma - 2(1-\gamma^2)] \tau^2 a \hat{\omega}_{1,xx} + 4D(1-\gamma^2 + \frac{\xi^2}{4}) \frac{1}{\alpha} \hat{\omega}_1 = \\ = D\alpha \hat{p}_2 - \tau^2 a^3 \hat{p}_{2,xx}. \end{aligned} \quad (170)$$

After applying the Laplace transform with respect to the reduced time

$$\hat{\omega}^*(x, p) = \int_{0^-}^{\infty} \hat{\omega}(x, \xi) e^{-p\xi} d\xi,$$

(169) reads

$$p D^* \hat{\omega}_2^* = \frac{1-\gamma^2}{2(1-\gamma^2 + \frac{\xi^2}{4})} \alpha_0 p D^* \hat{\theta}^* \quad (171)$$

and has the solution

$$\hat{\omega}_2 = \frac{1-\gamma^2}{2(1-\gamma^2 + \frac{\xi^2}{4})} \alpha_0 \theta, \quad (172)$$

which represents the solution for uniform heating. Applying the Laplace

transform to (170)

$$\begin{aligned} a^3 \tau^2 p D^* \hat{\omega}_{1,xxxx} + 2[p D^* \gamma - 2(1-\gamma^2)] \tau^2 a \hat{\omega}_{1,xx} + \\ + 4 p D^* (1-\gamma^2 + \frac{\xi^2}{4}) \frac{1}{\alpha} \hat{\omega}_1^* = p D^* \alpha \hat{p}_2^* - \tau^2 a^3 \hat{p}_{2,xx}^*, \end{aligned} \quad (173)$$



and subsequently the Fourier transform,

$$\left\{ a^3 \tau^2 p D^* s^4 + 2 [p D^* \nu - 2(1-\nu^2)] \tau^2 a s^2 + 4 p D^* (1-\nu^2 + \frac{\tau^2}{4}) \frac{1}{a} \right\} \hat{\omega}_1^{**} =$$

$$= (p D^* a - \tau^2 a^3 s^2) \hat{P}_2^{**} , \quad (174)$$

gives an algebraic equation for the deflection. With the transform of the ringload

$$P_2 = P_0 H(t) \delta(x) , \quad \hat{P}_2^{**} = \frac{P_0}{p} ,$$

(174) can be solved for the deflection:

$$\frac{\hat{\omega}_1^{**}}{a^3 P_0} = - \frac{(a^2 s^2 - \frac{p D^*}{\tau^2}) \frac{1}{p}}{p D^* a^4 s^4 + 2 [p D^* \nu - 2(1-\nu^2)] a^2 s^2 + p D^* [1 + \frac{1-\nu^2}{\tau^2/4}]} . \quad (175)$$

We now specialize the operator D to that of a standard solid (see Fig. 6):

(relaxation modulus of a standard solid)

$$D^* = \frac{h}{G' h'} \frac{G_2}{G_1 + G_2} \left( \frac{G_1}{p} + \frac{G_2}{p + \frac{G_1 + G_2}{\eta}} \right) ,$$

where  $\eta$  = viscosity ,

$$\tau = \frac{G_1 + G_2}{\eta} = \text{relaxation time} .$$

Assuming

$$\frac{G_1}{G_2} = \frac{1}{4} , \quad \frac{h}{h'} = 40 , \quad \frac{G_1}{G'} = \frac{2.6}{400} , \quad \frac{G_2}{G'} = \frac{2.6}{100} , \quad (176)$$

$$\frac{h}{a} = \tau = \frac{1}{50} , \quad \nu = 0.3 ,$$

we find

$$D^* = \frac{0.208}{p} + \frac{0.832}{p + \frac{1}{\tau}} ,$$

and after introduction of a dimensionless coordinate  $\eta = as$ , (175) becomes:

$$\frac{\hat{\omega}_1^{**}}{a^3 P_0} = - \frac{\eta^2 - 2600}{1.04(\eta^4 - 2.9\eta^2 + 9101)} \cdot \frac{P + 0.2 \frac{1}{\tau}}{P \left[ P + \frac{\eta^4 - 16.9\eta^2 + 9101}{\eta^4 - 2.9\eta^2 + 9101} 0.2 \frac{1}{\tau} \right]} \quad (177)$$

The inverse transform was found in [23], pg. 299:

$$\begin{aligned} \frac{\hat{\omega}_1^*}{a^3 P_0} = & - \frac{\eta^2 - 2600}{1.04(\eta^4 - 2.9\eta^2 + 9101)} \frac{\eta^4 - 2.9\eta^2 + 9101}{(\eta^4 - 16.9\eta^2 + 9101)} \frac{\tau}{0.2} \left\{ 0.2 \frac{1}{\tau} - \left( 0.2 \frac{1}{\tau} - \right. \right. \\ & \left. \left. - 0.2 \frac{1}{\tau} \frac{\eta^4 - 16.9\eta^2 + 9101}{\eta^4 - 2.9\eta^2 + 9101} \right) \exp \left( - \frac{\eta^4 - 16.9\eta^2 + 9101}{\eta^4 - 2.9\eta^2 + 9101} 0.2 \frac{1}{\tau} \right) \right\}. \end{aligned} \quad (178)$$

Observing that  $\hat{w}_1$  is an even function in  $\eta$  it is seen that

$$\frac{\hat{\omega}_1}{a P_0} = \frac{2}{2\pi} \int_0^\infty \left( \frac{\hat{\omega}_1^*}{a^3 P_0} \right) \cos \left( \eta \frac{x}{a} \right) d\eta. \quad (179)$$

Expanding the exponential function

$$\exp \left( - \frac{\eta^4 - 16.9\eta^2 + 9101}{\eta^4 - 2.9\eta^2 + 9101} 0.2 \frac{1}{\tau} \right), \quad (180)$$

in a power series and comparing the integrals which result from substituting (178) and the power series of (180) into (179), it is seen that in approximating (180) by  $\exp(-0.2 \xi/\tau)$  only a term of order  $10^{-3}$  as compared to one is being neglected when  $\xi/\tau \leq 15$ . With this approximation (178) reduces to

$$\begin{aligned} \frac{\hat{\omega}_1^*}{a^3 P_0} = & - \frac{\eta^2 - 2600}{1.04(\eta^4 - 16.9\eta^2 + 9101)} (1 - e^{-0.2 \xi/\tau}) - \\ & - \frac{\eta^2 - 2600}{1.04(\eta^4 - 2.9\eta^2 + 9101)} e^{-0.2 \xi/\tau}. \end{aligned} \quad (181)$$

The inverse Fourier transform of (181) was found completely analogous to that of the first example (165) and the intermediate steps are therefore omitted.

Thus we obtain the part of the deflection due to the ringload:

$$\begin{aligned} \frac{\hat{w}_1}{a^2 P_0} = & \frac{\sqrt[4]{C}}{1.04} \left\{ \frac{\exp\left[-\frac{1}{2}\sqrt{2\sqrt{C}-B'}\frac{x}{a}\right]}{\sqrt{4C-B'^2}} \left[ \frac{2600}{\sqrt[4]{C}} \cos\left[\frac{\phi'}{2} - \frac{1}{2}\sqrt{2\sqrt{C}+B'}\frac{x}{a}\right] - \cos\left[\frac{\phi'}{2} + \frac{1}{2}\sqrt{2\sqrt{C}+B'}\frac{x}{a}\right] \right] \cdot \right. \\ & \cdot \left( 1 - e^{-\alpha^2 \xi/\tau} \right) + \frac{\exp\left[-\frac{1}{2}\sqrt{2\sqrt{C}-B''}\frac{x}{a}\right]}{\sqrt{4C-B''^2}} \left[ \frac{2600}{\sqrt[4]{C}} \cos\left[\frac{\phi''}{2} - \frac{1}{2}\sqrt{2\sqrt{C}+B''}\frac{x}{a}\right] - \right. \\ & \left. \left. - \cos\left[\frac{\phi''}{2} + \frac{1}{2}\sqrt{2\sqrt{C}+B''}\frac{x}{a}\right] \right] e^{-0.2 \xi/\tau} \right\}. \end{aligned} \quad (182)$$

The first part of (182) represents the long term solution, the second part the instantaneous deflection. A time profile at the origin and a longitudinal profile at  $\xi/\tau = 0.1$  were computed. Using the assumptions (176) we find

$$\begin{aligned} \sqrt[4]{C} &= 9.77, \quad \sqrt{2\sqrt{C}-B'} = 13.18, \quad \sqrt{2\sqrt{C}+B'} = 14.11, \quad \sqrt{2\sqrt{C}-B''} = 13.71, \\ \sqrt{2\sqrt{C}+B''} &= 13.92, \quad \frac{\phi'}{2} = 0.7408, \quad \frac{\phi''}{2} = 0.7778. \end{aligned}$$

Table of the time profile (plot see Fig. 7):

$\xi/\tau$	0	0.1	0.2	0.5	1.0	2.0	5.0	10	20	50	$\infty$
$\hat{w}/a^2 P_0$	9.297	9.304	9.311	9.332	9.365	9.420	9.533	9.620	9.664	9.670	9.677

It is seen from Fig. 7 that the simplification of the result for  $\xi/\tau > 15$  is fully justified. Table of the longitudinal profile (plot see Fig. 8):

$x/a$	0	0.05	0.1	0.2	0.5
$\hat{w}/a^2 P_0$	9.304	8.445	6.582	2.724	-0.385

We can see from the time profile that the influence of the viscoelastic core on the deflection is about 4%. It can be concluded therefore that the influence of the viscoelasticity of the core can be neglected in the computation of stresses and deformations of a sandwich structure. This is only valid within the scope of the above assumptions for the geometric and material properties. This statement is not valid for very weak cores as the assumption of small deflections is void in this case. Furthermore, the result would be expected to depend on the magnitude and time dependence of the transverse relaxation modulus of the core, which in this case is assumed to be infinite.

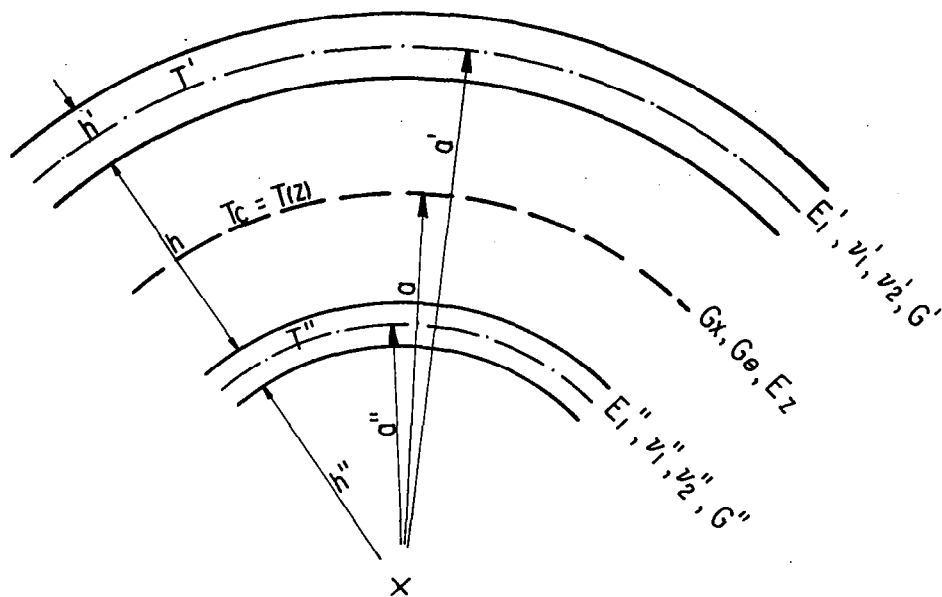


Fig. 1. Cross-Section of a Cylindrical Sandwich Shell

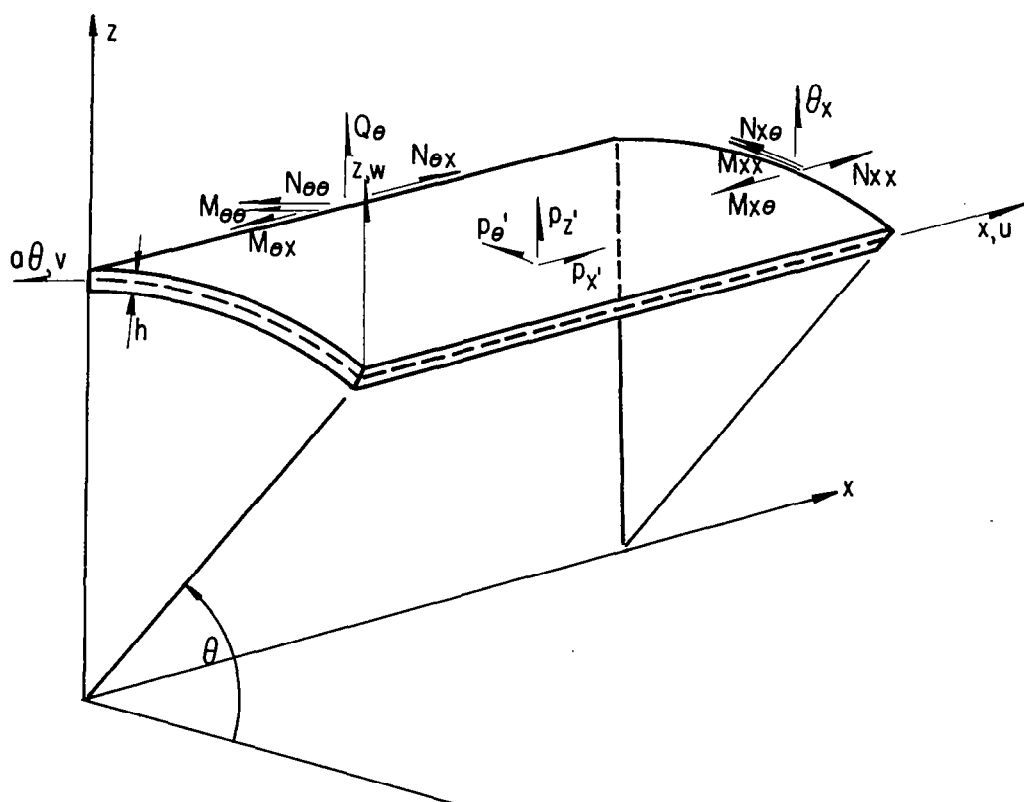


Fig. 2. Composite Shell Element

The elasticity sign convention is adopted. (Differs from Timoshenko's and Girkmann's sign convention.)

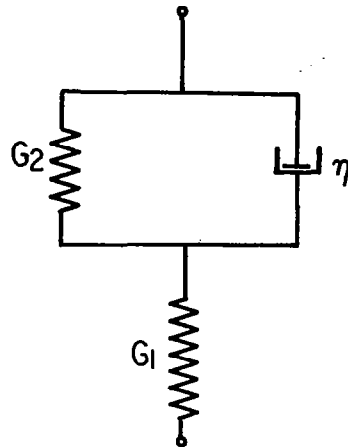


Fig. 6. Viscoelastic Core Behavior (Standard Solid)

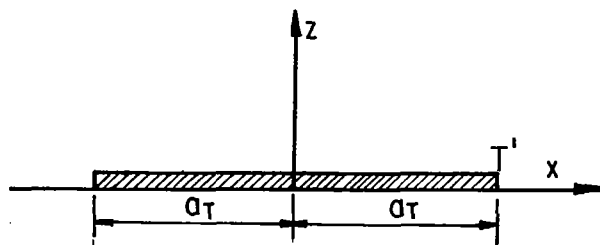


Fig. 3. Temperature Distribution on the Shell

Spatial distribution of temperature on an elastic sandwich cylinder. The material properties are independent of temperature.

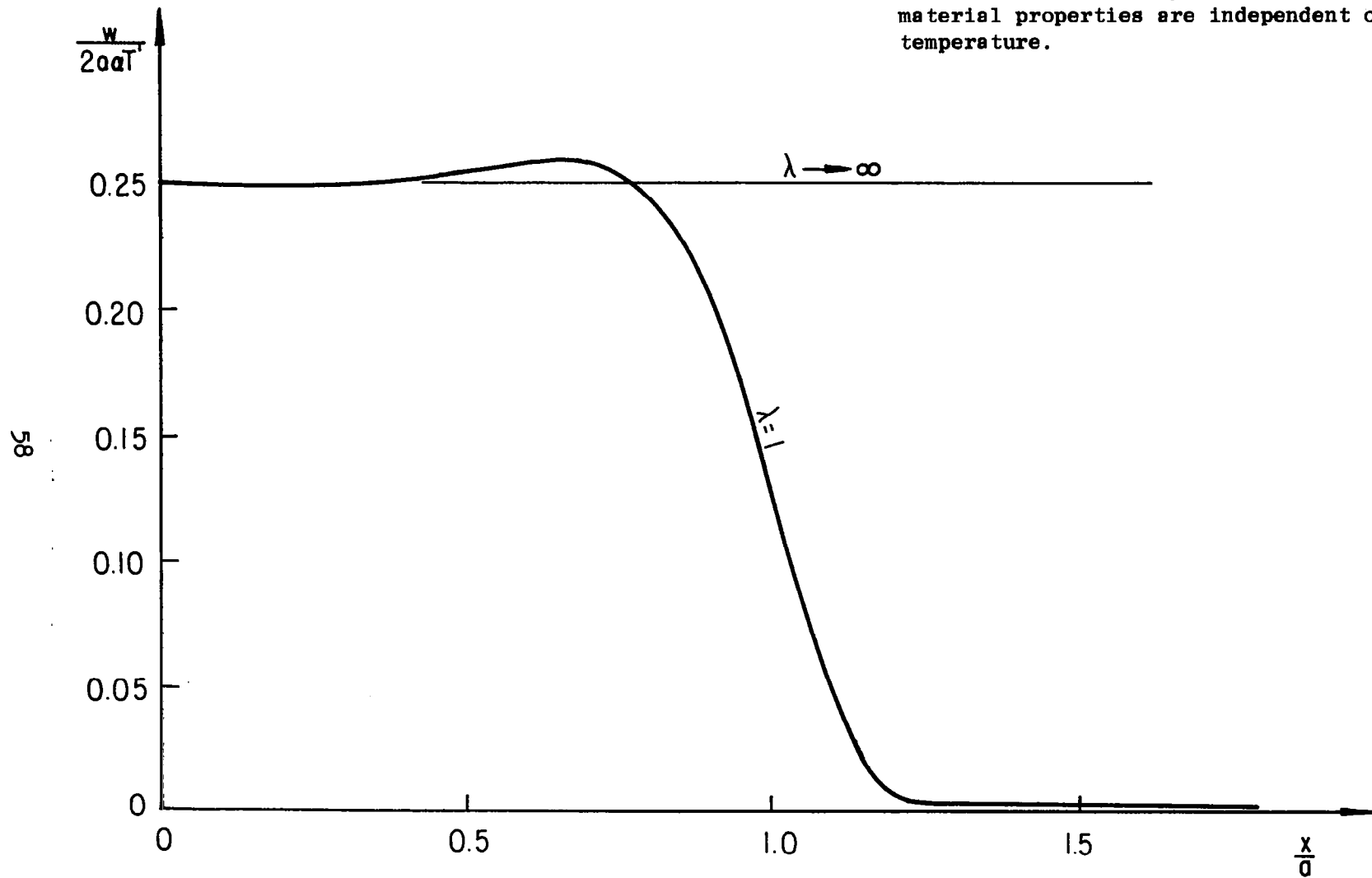


Fig. 4. Longitudinal Profile of the Transverse Deflection

Ring heating of an elastic sandwich cylinder.  
The material properties are independent of  
temperature.

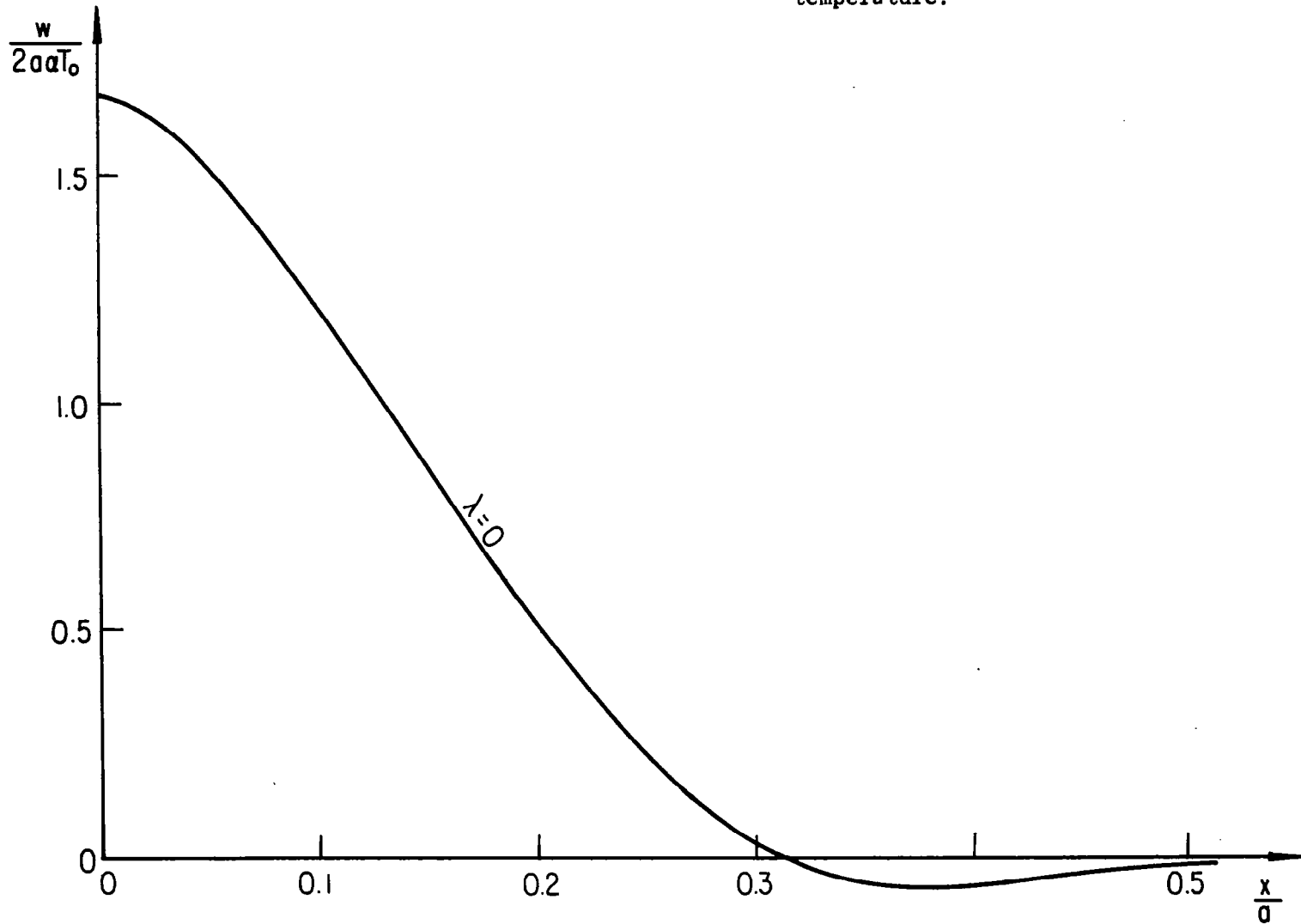


Fig. 5. Longitudinal Profile of the Transverse Deflection



Uniform heating of a sandwich cylinder in addition to a ringload. The viscoelastic core has temperature dependent properties.

Time profile of the deflection due to the ringload at the origin.

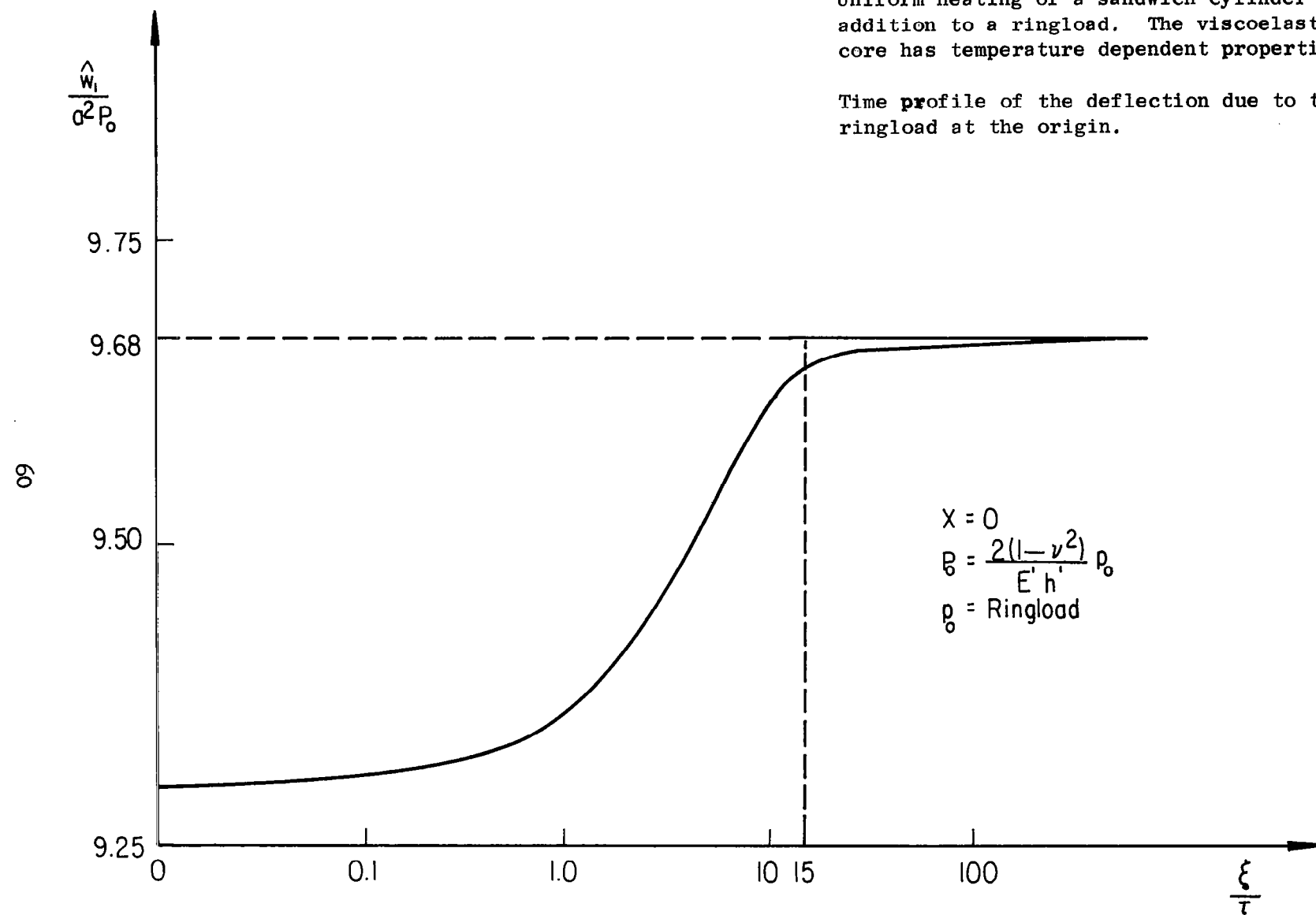


Fig. 7

Uniform heating of a sandwich cylinder in addition to a ringload. The viscoelastic core has temperature dependent properties.

Longitudinal profile of the transverse deflection due to the ringload at the time  $\xi = 0.1 \tau$ .

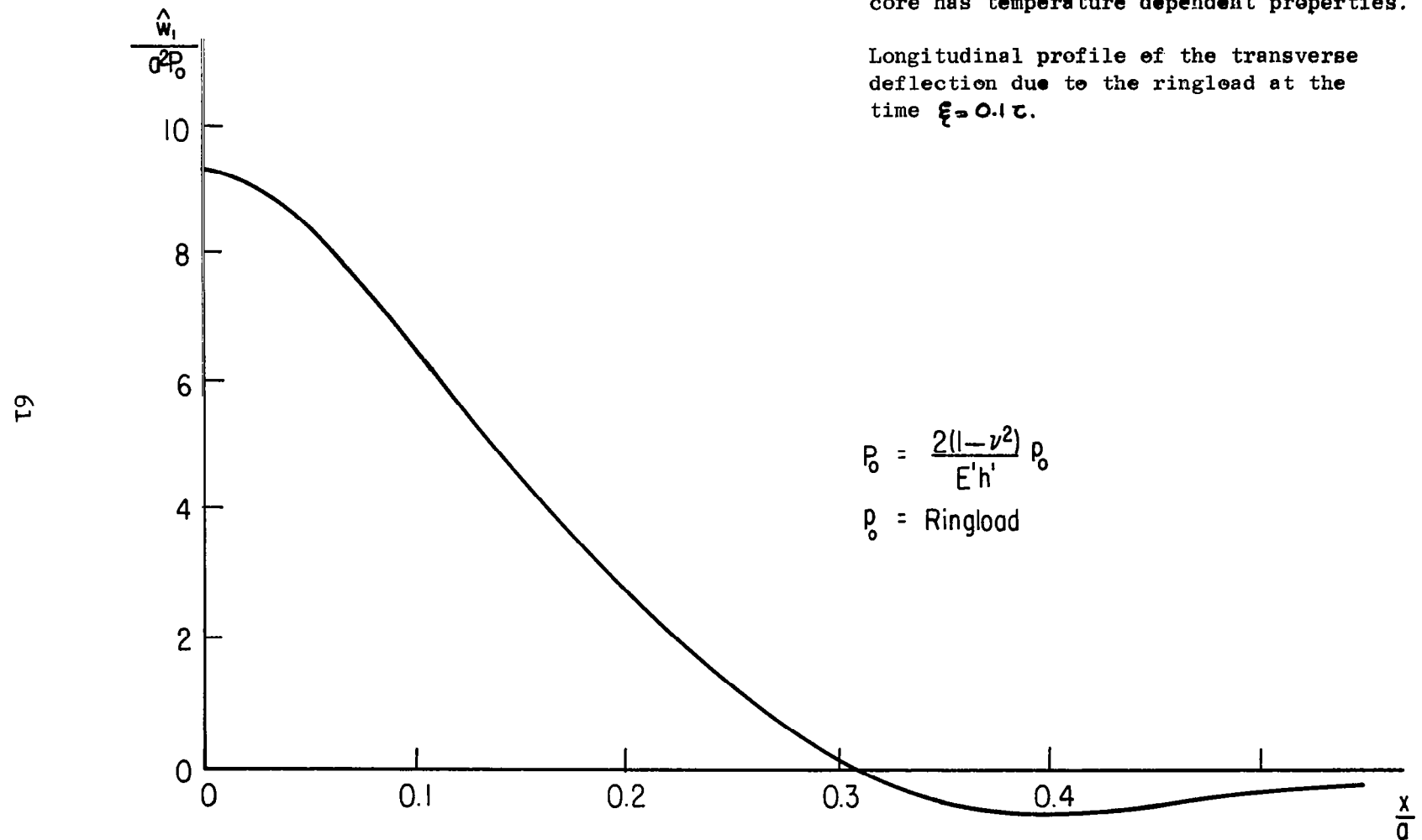


Fig. 8

### List of Symbols

$a$	radius of cylinder
$a_T$	heated length of cylinder
$l$	length of cylinder
$m_i$	surface couples
$n_i$	direction cosine at the boundary
$p$	variable in Laplace space
$p_i$	surface loads
$s$	contour coordinate
	also: variable in Fourier space
$t$	time
$u_i$	displacement generally
$u$	displacement in x-direction
$v$	displacement in $\theta$ -direction
$w$	displacement in z-direction
$x$	longitudinal coordinate
$y$	second plate coordinate
$z$	transverse coordinate
$\alpha_{1,2}$	$\alpha_1, \alpha_2$

$C_{ij}$	material constants of orthotropic media
$D$	$\frac{2G_x h(1-\nu_1'\nu_2')}{E'h'}$
$E$	$\frac{2G_\theta h(1-\nu_1'\nu_2')}{E'h'}$
$E', "$	Young's modulus of the facings
$E_z$	transverse compression modulus of core
$F$	$\frac{E_z h(1-\nu_1'\nu_2')}{E'h'}$
$G_{1,2}$	$\gamma' \pm \gamma''$
	also: elastic moduli of standard solid
$G', "$	shear modulus of the facings
$G_{x,\theta}$	transverse shear modulus in the core
$M_i$	$2m_i(1-\nu_1'\nu_2')/E'h'$
$P_i$	$2p_i(1-\nu_1'\nu_2')/E'h'$
$S_1$	part of the surface where the stresses are prescribed
$T$	temperature
$T_m$	$T'+T''$
$T$	$T'-T''$
$T_i$	traction at the boundary
$T_o$	reference temperature
$U$	strain energy
$V$	volume
$W$	external work

$\alpha_{ij}$	coefficient of linear temperature expansion also: angle of opening of the cylinder
$\alpha_0\theta$	pseudo temperature
$\beta$	ratio of the extensional stiffnesses of the facings
$\nu$	ratio of shear stiffness to extensional stiffness
$\delta_{ij}$	Kronecker symbol
$\epsilon_{ij}$	strain tensor
$\eta$	dimensionless variable in the Fourier space also: viscosity of dashpot
$\theta$	circumferential coordinate
$\chi$	third term in the expansion for $\underline{w}$
$\lambda$	ratio of heated length to radius of cylinder
$\mu_i$	Lagrange multipliers
$\nu$	Poisson's ratio
$\xi$	reduced time
$\tau$	ratio of core thickness to radius also: relaxation time of standard solid
$\phi$	second term in the expansion for $\underline{u}$ also: shift function for a thermorheologically simple material
$\chi$	second term in the expansion for $\underline{v}$
$\psi$	second term in the expansion for $\underline{w}$
$\underline{\dots}$	indicates that displacement depends on all 3 coordinates also: indicates that stress resultant is expressed in terms of the displacements

'... upper facing  
also: indicates integration variable when with time or temperature

"... lower facing

\*... refers to Laplace space or Fourier space

^... indicates that a function is expressed in reduced time

-... indicates a prescribed function

1... indicates sum of 2 ratios  
also: quantity in x-direction

2... indicates difference of 2 ratios  
also: quantity in  $\theta$ -direction

...<sup>m</sup> mean value

...<sup>(i)</sup> mean quantity of i-th order

c... core

o... ring quantity (load or temperature)

Some of the foregoing symbols are used in a different meaning for the two examples. Since they are clearly defined and only temporarily introduced the reader will not encounter any ambiguities.

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